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On generalized quasi-conformal like recurrent manifolds

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Abstract. The object of the present paper is to introduce a generalized class of recurrent manifold called quasi-conformal like recurrent manifolds. Some geometric properties of generalized quasi-conformal like recurrent manifolds are obtained. The application of such a manifold in general relativity are investigated.

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Key words and phrase: recurrent manifold, generalized recurrent manifold, generalized quasi-conformal curvature tensor, quasi-conformal like curvature tensor, scalar curvature.

Introduction: It is well known that symmetric spaces play a great role in the differential geometry. The study of Riemann spaces was initiated by E. Cartan [8] . According to E. Cartan a Riemannian manifold is said to be locally symmetric if its Riemannian curvature tensor R of type (0, 4) satisfies the condition $\nabla R = 0$ where ∇ denote the Levi-Civita connection. Later many authors have weakened this notion in different ways such as recurrent manifold by Walker [25], generalized recurrent by Dubey [12], quasi-generalized recurrent by Shaikh and Roy [21], hyper-generalized recurrent by Shaikh and Patra [20], semisymmetric manifold by Cartan [8], pseudosymmetric manifolds by Chaki [9], pseudosymmetric manifolds by Deszcz[11], weakly symmetric manifold by Tamássy and Binh [23]. Thereafter, a lot of research has been carried out in weakly symmetric manifold. For details, we refer to [1], [2], [3], [4], [5], [7] and the references there in.

In tune with Yano and Sawaki [24], recently Baishya and Chowdhury [6] introduce a new type of tensor field, named generalized quasi-conformal curvature tensor. The beauty of generalized quasi-conformal curvature tensor lies in the fact that it has the flavor of Riemannian curvature R, conformal curvature tensor C, conharmonic curvature tensor $\bar{C}$, concircular curvature tensor E, projective curvature tensor P and m-projective curvature tensor H as special cases. The generalized quasi-conformal curvature tensor is defined as

$$\widetilde{W}(X,Y)Z = \frac{n-2}{n}[\{1-b+(n-1)a\} - \{1+(n-1)(a+b)c]C(X,Y)Z$$

$+[1-b(n-1)a]E(X,Y)Z+(n-1)(b-a)P(X,Y)Z+\frac{n-2}{n}(c-1)\{1+(n-1)(a+b)\}\bar{C}(X,Y)Z$ (1.1)

For all X, Y, $Z \in \chi(M^n)$, the set of all vector field of the manifold M, where a, b and c are real constants. The generalized quasi-conformal curvature tensor $\widetilde{W}$ is reduced to be (1) Riemannian curvature tensor R, if a=b=c=0, (2) conformal curvature tensor C if $a=b=-\frac{1}{n-2}$, c=1, (3) conharmonic curvature tensor $\bar{C}$, if $a=b=-\frac{1}{n-2}$, c=0, (4) concircular curvature tensor E, if a=b=0 and c=1, (5) projective curvature tensor P, if a= $-\frac{1}{n-1}$, b=0, c=0 and (6) M-projective curvature tensor H, if a=b= $-\frac{1}{2(n-1)}$ and c=0, is introduced by G. P. Pokhariyal and R. S. Mishra [17], which is defined as follows

$$H(X,Y)Z=R(X,Y)Z-\frac{1}{2(n-1)}[S(Y,Z)X-S(X,Z)Y+g(Y,Z)QX-g(X,Z)QY],$$

Where Q be the symmetric endomorphism of the tangent bundle corresponding to the Ricci tensor S.

For our convenience in this paper we renamed generalized quasi-conformal curvature $\widetilde{W}$ as quasi-conformal like curvature tensor.

Generalizing the notion of recurrent manifold, in this paper we introduce the notion of generalized quasi-conformal like recurrent manifold. A Riemannian manifold M, n>2, is said to be generalized quasi-conformal like recurrent manifold if its quasi-conformal like curvature tensor $\widetilde{W}$ satisfies the following relation:

$$\left(\nabla_X\widetilde{W}\right)(Y,U,V,W)=A(X)\widetilde{W}(Y,U,V,W)+\alpha(X)G(Y,U,V,W). \qquad (1.2)$$

Where $G(Y,U,V,W)=[g(U,V)g(Y,W)-g(Y,V)g(U,W)]$

And A, α are non-zero 1-forms which is defined as $A(X)=g(X,\theta)$, $\alpha=g(X,\emptyset)$. Such an n-dimensional manifold is denoted by $(GQCLK)_n$.

The beauty of such $(GQCLK)_n$ space is that for suitable choice of 1-forms and the real constants a, b, and c we get different spaces such as

(i) Generalized recurrent manifolds [12] (for a=b=c=0),

(ii) Generalized conformally recurrent manifolds [16] (for a=b= $-\frac{1}{n-2}$, c=1),

(iii) Generalized conharmonically recurrent manifolds [20](for a=b= $-\frac{1}{n-2}$, c=0)

(iv) Generalized concircularly recurrent manifolds[13](for a=b=0 and c=1)

(v) Generalized M-projectively recurrent manifolds[15] (for a=b= $-\frac{1}{2(n-1)}$ and c=0)

(vi) Recurrent manifolds[25] (for a=b=c=$\alpha = 0$)
(vii) Symmetric manifolds[8] (for a=b=c=$A = \alpha = 0$).

We organized this paper as follows; section 2 is concerned with some basic geometric properties of $(GQCLK)_n$. In section 3 we find a sufficient condition for a $(GQCLK)_n$ to be a quasi-Einstein manifold. In the last section we studied some application of such manifold.

2. Some geometric properties of $(GQCLK)_n$.

In this section, we consider a Riemann manifold (M^n, g) n>2 which is generalized quasi-conformal like recurrent manifold. Now, contracting (1.2) we have

(2.1) $(\bar{b}+\frac{n}{n-1}\bar{c})(\nabla_X S)(Y,W) - \left(\frac{\bar{d}}{n-2} + \frac{\bar{b}}{n} + \frac{\bar{c}}{n-1}\right) dr(X)g(Y,W) = A(X)[\left(\bar{b} + \frac{n}{n-1}\bar{c}\right) \mathrm{S(Y,W)} - \left(\frac{\bar{d}}{n-2} + \frac{\bar{b}}{n} + \frac{\bar{c}}{n-1}\right) rg(Y,W)] + (n-1)\alpha(X)g(Y,W)$.

Where $\bar{a} = \frac{n-2}{2}[\{1-b+(n-1)a\} - \{1+(n-1)(a+b)\}c]$, $\bar{b} = [1-b+(n-1)a]$,

$\bar{c} = (n-1)(b-a)$ and $\bar{d} = \frac{n-2}{2}(c-1)\{1+(n-1)(a+b)\}$.

On simplification we get

(2.2) $(\bar{b}+ \frac{n}{n-1}\bar{c})\ (\nabla_X S)(Y,W\ ) = [A(X)]\left(\bar{b} + \frac{n}{n-1}\bar{c}\right) S(Y,W) + \left[(n-1)\alpha(X) - \left(rA(X) - dr(X)\right)\left(\frac{\bar{d}}{n-2} + \frac{\bar{b}}{n} + \frac{\bar{c}}{n-1}\right)\right] g(Y,W)$.

This leads the following:

Theorem 2.1. A $(GQCLK)_n$ is a generalized Ricci recurrent manifold [14] provided (n-1) $\bar{b}$ + n $\bar{c} \neq 0$.

Taking contraction over Y and W in (2.1) we obtain

(2.3) $-\frac{n}{n-2}\bar{d}\ dr(X) = -\frac{n}{n-2}\bar{d}\ r\ A(X) + (n-1)n\alpha(X)$, i.e.,

(2.4) $dr(X) = r\ A(X) - \frac{(n-1)(n-2)}{\bar{d}}\ \alpha(X)$.

If we take the scalar curvature of $(GQCLK)_n$ space is non-zero constant, then we have $dr(X)$=0 and from (2.4) we have

(2.5) $r\ A(X) - \frac{(n-1)(n-2)}{\bar{d}}\ \alpha(X) = 0$

This gives the following:

Theorem 2.2. If the scalar curvature of a $(GQCLK)_n$ is non-zero constant, then the 1-forms are related by the expression (2.5) .

Theorem 2.3. If the scalar curvature of a $(GQCLK)_n$ is non-zero constant, then the 1-forms are co directional provided $\bar{d} \neq 0$.

Again contraction of (2.1) over X and W yields

(2.6) $\left[\frac{n-2}{2}\left(\bar{b} + \frac{n}{n-1}\bar{c}\right) - \frac{\bar{d}}{n-2}\right] dr(Y) = (\bar{b}+ \frac{n}{n-1}\bar{c})A(QY) - \left(\frac{\bar{d}}{n-2} + \frac{\bar{b}}{n} + \frac{\bar{c}}{n-1}\right) rA(Y) + (n-1)\alpha(Y)$,

where $A(QX) = g(QX,\theta)$.

Using (2.4) in (2.6) we have

(2.7) $r = \frac{2A(QX)}{A(X)} + \frac{2(n-1)(n-2)^2\ \alpha(X)}{A(X)n\ \bar{d}}$ provided $(n-1)\bar{b} + n\bar{c} \neq 0$.

Thus we can state the following:

Theorem 2.4. In a $(GQCLK)_n$ the scalar curvature exist provided $(n-1)\bar{b} + n\bar{c} \neq 0$, $\bar{d} \neq 0$ and is given by (2.7).

Again if the scalar curvature $r = 0$ then we have

(2.8) $A(QY) = -\frac{(n-1)(n-2)^2\ \alpha(Y)}{A(X)n\ \bar{d}}$

Theorem 2.5. If the scalar curvature of a $(GQCLK)_n$ is vanishes, then the 1-forms are related by the expression (2.8) .

If a $(GQCLK)_n$ is a Ricci-symmetric, then we have $\nabla S=0$, $dr = 0$ and r is constant. So from (2.2) we get

(2.9) $$A(X)\left(\bar{b} + \frac{n}{n-1}\bar{c}\right)S(Y,W) + \left[(n-1)\alpha(X) - rA(X)\left(\frac{\bar{d}}{n-2} + \frac{\bar{b}}{n} + \frac{\bar{c}}{n-1}\right)\right]g(Y,W) = 0.$$

On simplification we have

(2.10) $$S(Y,W) = -\frac{\left[(n-1)\alpha(X) - rA(X)\left(\frac{\bar{d}}{n-2} + \frac{\bar{b}}{n} + \frac{\bar{c}}{n-1}\right)\right]g(Y,W)}{A(X)\left(\bar{b} + \frac{n}{n-1}\bar{c}\right)}$$ provided $(n-1)\bar{b} + n\bar{c} \neq 0$.

This can be written as

$S(Y, W) = \mu\ g(Y, W)$, where $\mu = -\frac{\left[(n-1)\alpha(X) - rA(X)\left(\frac{\bar{d}}{n-2} + \frac{\bar{b}}{n} + \frac{\bar{c}}{n-1}\right)\right]}{A(X)\left(\bar{b} + \frac{n}{n-1}\bar{c}\right)}$, a scalar.

Thus we can state the following:

Theorem 2.6. A Ricci-symmetric $(GQCLK)_n$ is a Einstein manifold provided $(n-1)\bar{b} + n\bar{c} \neq 0$.

3. Sufficient condition for a generalized quasi-conformal like recurrent manifold to be a quasi Einstein manifold

In a $(GQCLK)_n$ the vector field θ defined by $g(X,\theta) = A(X)$ for any vector field X is said to be a concircular vector field [19] if the following equation is satisfied

(3.1) $(\nabla_X A)(Y) = \beta\ g(X, Y) + \varphi\ (X)\ A(Y)$, where β is a non-zero scalar and φ is a closed 1-form. If θ is a unit one, then (3.1) becomes

(3.2) $(\nabla_X A)(Y) = \beta\ [g(X, Y) - A(X)\ A(Y)]$.

We consider that $(GQCLK)_n$ admits a unit concircular vector field defined by (3.1), where β is a non-zero constant. Proceeding in the same way in [15] in Riemann case we get the flowing result:

(3.3) $S(X,\theta) = (n-1)\beta^2 A(X)$.

(3.4) $(\nabla_X S)(Y,\theta) = (n-1)\beta^2(\nabla_X A)(Y) - S(Y,\nabla_X\theta)$.

(3.5) $S(Y,\nabla_X\theta) = \beta[S(X,Y) - A(X)S(Y,\theta)]$.

Using (3.2) in (3.4) we have

(3.6) $(\nabla_X S)(Y,\theta) = (n-1)\beta^3[g(X,Y) - A(X)A(Y)] - S(Y,\nabla_X\theta)$.

By virtue of (3.5) it follows from (3.6) that

(3.7) $(\nabla_X S)(Y,\theta) = (n-1)\beta^3[g(X,Y) - A(X)A(Y)] - \beta[S(X,Y) - A(X)S(Y,\theta)]$.

Using (3.3) in (3.7) we get

(3.8) $(\nabla_X S)(Y,\theta) = (n-1)\beta^3 g(X,Y) - \beta S(X,Y)$.

From (2.2) we have

(3.9) $(\bar{b}+ \frac{n}{n-1}\bar{c})\ (\nabla_X S)(Y,\theta) = [A(X)]\left(\bar{b} + \frac{n}{n-1}\bar{c}\right)S(Y,\theta) + \left[(n-1)\alpha(X) - \left(rA(X) - dr(X)\right)\left(\frac{\bar{d}}{n-2} + \frac{\bar{b}}{n} + \frac{\bar{c}}{n-1}\right)\right]g(Y,\theta)$.

Now using (3.8) and (3.3) in (3.9) we get

(3.10) $(\bar{b}+ \frac{n}{n-1}\bar{c})[(n-1)\beta^3 g(X,Y) - \beta S(X,Y)] = A(X)\left(\bar{b} + \frac{n}{n-1}\bar{c}\right)(n-1)\beta^2 A(Y) + \left[(n-1)\alpha(X) - \left(rA(X) - dr(X)\right)\left(\frac{\bar{d}}{n-2} + \frac{\bar{b}}{n} + \frac{\bar{c}}{n-1}\right)\right]A(Y)$.

Now if the scalar curvature is constant, then *dr* = 0 and using (2.5) in (3.10)

(3.11) $S(X,Y) = \beta^2(n-1)g(X,Y) + \frac{\frac{r}{n} - (n-1)\beta^2}{\beta}A(X)A(Y)$ provided (n-1) $\bar{b}$ + n $\bar{c}$ $\neq 0$.

Since β is non-zero constant (3.11) can be written as

$S(X,Y) = l\ g(X,Y) + m\ A(X)A(Y)$ provided (n-1) $\bar{b}$ + n $\bar{c} \neq 0$.

Where $l = \beta^2$(n-1) and $m = \frac{\frac{r}{n} - (n-1)\beta^2}{\beta}$ are two non-zero constant as β is non-zero constant. Hence the manifold is a quasi-Einstein manifold. This leads us the following:

Theorem 3.1. A $(GQCLK)_n$ with constant scalar curvature becomes a quasi Einstein manifolds if the associated vector field θ defined by $g(X,\theta) = A(X)$ is a unit one and if it is a unit concircular vector field whose associated scalar is a non-zero constant.

4. Applications of perfect fluid Ricci symmetric $(GQCLK)_n$ space time

This section deals with $(GQCLK)_n$ spacetime. A spacetime is a connected 4−dimensional Lorentzian manifold. Hence a generalized quasi conformal like recurrent spacetime is a 4−dimensional connected Lorentzian generalized quasi conformal like recurrent manifold in which the vector field θ associated to the 1-form A is a unit timelike vector field. Such spacetimes is called $(GQCLK)_n$ spacetimes. Here we consider a special type of spacetime which is called Ricci symmetric generalized quasi conformal like recurrent spacetime. A semi-Riemannian four-dimensional Ricci symmetric generalized quasi-conformal like recurrent manifold may similarly be defined by taking a Lorentz metric g with signature (+,+,+,-). In this case we consider a Ricci symmetric $(GQCLK)_n$ spacetime with the timelike velocity vector field $g(\theta,\theta) = -1$. So, Theorem 2.6. will also hold in such a spacetime. The Einstein's equation without cosmological constant is

(4.1) $S(X,Y) - \frac{r}{2} g(X,Y) = k\, T(X,Y).$

Where k is the gravitational constant, T is the energy- momentum tensor of type(0,2). We consider a perfect fluid spacetime. Then the energy- momentum tensor is of the form

(4.2) $T(X,Y) = (\sigma + \rho)A(X)A(Y) + \rho\, g(X,Y)$,

where σ and ρ as the energy density and isotropic pressure of the fluid respectively. By virtue of (4.2), (4.1) can be written as

(4.3) $S(X,Y) - \frac{r}{2} g(X,Y) = k\,[\,(\sigma + \rho)A(X)A(Y) + \rho\, g(X,Y)]$

Taking contraction (4.3) over X and Y we get

(4.4) $r = k(\sigma - 3\rho).$

Now setting $W = X = \theta$ and using (2.5) in (2.10) for n=4 we get

(4.5) $S(Y,\theta) = \dfrac{3\,\alpha(\theta)\,[1 - \frac{2}{\bar{d}}(\frac{\bar{d}}{2} + \frac{\bar{b}}{4} + \frac{\bar{c}}{3})]\,A(Y)}{(\bar{b} + \frac{4}{3}\bar{c})}$

Again substituting $Y = \theta$ in (4.3) we get

(4.6) $S(X,\theta) = \frac{r}{2}\, A(X) - k\,\sigma\, A(X).$

By virtue of (4.4) and (4.5) , (4.6) takes the form

(4.7) $\alpha(\theta) = -\dfrac{k(\sigma+3\rho)(\bar{b}+\frac{4}{3}\bar{c})}{6[1 - \frac{2}{\bar{d}}(\frac{\bar{d}}{2} + \frac{\bar{b}}{4} + \frac{\bar{c}}{3})]}$

Again applying (4.7) in (4.5) we get

(4.8) $S(Y,\theta) = -\frac{k(\sigma+3\rho)}{2}\, A(Y)$

Putting $Y = \theta$ in (4.8) we have

(4.9) $S(\theta,\theta) = \frac{k(\sigma+3\rho)}{2}$

From (4.3) and (4.4) we get

(4.10) $S(X,Y) = k\,[\,(\sigma+\rho)A(X)A(Y) + \frac{(\sigma-\rho)}{2}\,g(X,Y)]$

And hence

(4.11) $S(QX,Y) = k\,[\,(\sigma+\rho)A(QX)A(Y) + \frac{(\sigma-\rho)}{2}\,S(X,Y)]$

Taking contraction (4.11) over X and Y we get

(4.12) $\|Q\|^2 = k\,[\,(\sigma+\rho)S(\theta,\theta) + \frac{(\sigma-\rho)}{2}\,r\,]$

Using (4.4) and (4.9) we obtain

(4.13) $\|Q\|^2 = k^2(\sigma^2+3\rho^2)$.

If we take the square of the length of the Ricci operator of the perfect fluid $(GQCLK)_n$ spacetime be $\frac{1}{3}r^2$, where r is the scalar curvature of the spacetime, then from (4.13), we have

(4.14) $\frac{1}{3}r^2 = k^2(\sigma^2+3\rho^2)$.

Which yields by virtue of (4.4) that

(4.15) $k^2(\sigma+3\rho)\sigma = 0$.

Since $\sigma+3\rho \neq 0$ and $k \neq 0$, it follows from (4.15) that $\sigma = 0$, which is not possible as when the pure matter exists, σ is always greater than zero. Hence the spacetime under consideration can not contain pure matter. Thus we can state the following:

Theorem 4.1. A perfect fluid Ricci symmetric $(GQCLK)_n$ spacetime satisfying Einstein's field equation without cosmological constant can not contain pure matter if the square of the length of the Ricci operator is $\frac{1}{3}r^2$.

We know that if the Ricci tensor S of type (0,2) of the spacetime satisfies the condition [18]

(4.16) $S(X,X) > 0$,

For every timelike vector field X, then (4.16) is called timelike convergence condition. Now we determine the sign of the pressure in such a spacetime without pure matter. Hence for $\sigma = 0$, (4.4) yields

(4.17) $r = -3\rho k$.

Hence from (4.2) we get

(4.18) $T(\theta,\theta) = \sigma = 0$.

Thus from (4.1) and (4.17) it follows that

$$\rho = \frac{2}{3k} S(\theta, \theta)$$

Since $S(\theta, \theta) > 0$, it follows from above relation that $\rho > 0$. Thus we can state the following:

Theorem 4.2. A perfect fluid Ricci symmetric $(GQCLK)_n$ spacetime having no pure matter, satisfying Einstein's field equation without cosmological constant and if its Ricci tensor follows the timelike convergence condition , the pressure of the fluid is positive.

References

[1] Baishya, K. K., *On generalized weakly symmetric manifolds*, Bulletin of the Transilvania University of Brasov, Series III:Mathematics, Informatics, Physics, 10(59) No. 1, (2017), 31--38.

[2] Baishya, K. K., *On generalized pseudo symmetric manifold*, submitted.

[3] Baishya, K. K., *On generalized semi-pseudo symmetric manifold*, submitted.

[4]Baishya, K. K., *Note On almost generalized pseudo-Ricci symmetric manifold*, Kyungpook Mathematical journal, 57(3) (2017), 517--523.

[5] Baishya, K. K., *On an almost generalized pseudo Ricci symmetric spacetime* , to appear in Italian journal of Pure and Applied Mathematics.

[6] Baishya, K. K., and Chowdhury, R. P., *On generalized Quasi-Conformal N(k, μ)-Manifolds*, Comm. Korean. Soc., 31 (1) (2016), 163--176.

[7] Baishya, K. K., Choudhury, P. R., Josef, M. and Peska, P., *On almost generalized weakly symmetric Kenmotsu Manifolds*, Acta. Univ, Palacki. olomue., Fac, rer. nat., Mathematica, 55(2) (2016), 5--15.

[8] Cartan, E., *Sur une classes remarquable d'espaces de Riemannian*, Bull. Soc. Math., France, 54 (1926), 214--264.

[9] Chaki, M. C., *On pseudosymmetric manifolds*, An. Sti. Ale Univ.,
``AL. I. CUZA" Din Iasi, 33 (1987), 53--58.

[10] Chaki, M. C. and Kawaguchi, T., *On almost pseudo Ricci symmetric manifolds*, Tensor N. S., 68(1) (2007), 10--14.

[11] Deszcz, R., *On pseudosymmetric spaces*, Bull. Soc. Math. Belg. Sér. A,
44(1) (1992), 1--34.

[12] Dubey, R. S. D., *Generalized recurrent spaces*, Indian. J . Pure Appl. Math., (10) (1979), 1508--1513.

[13] De, U. C. and Gazi, A. K., *On generalized concircularly recurrent manifolds*, Studia. Sci. Math. Hungar, 46(2) (2009), 287--296.

[14] De, U. C., Guha, N. and Kamilya, D., *On generalized Ricci recurrent manifolds*, Tensor, N. S., 56 (1995), 312--317.

[15] De, U. C. and Pal, P., *Ogeneralized M-projectively recurrent manifolds*, Ann.Univ. Paedagog. Crac.Stud. Math., 13 (2014), 287--296.

[16] Ozdemir, F. and Canfes, E., *On generalized conformally recurrent kaehlerian weyl spaces*, Iranian Journal of Science and Technology transaction a, science A3(3) (2012), 299--304.

[17] Pokhariyal,G.P., and Mishra, R. S., *Curvatur tensors and their relativistics significance*, Yokohama Math. J., 18 (1970), 105--108.

[18] Sach, R.K. and Hu, W., *General relativity for mathematician*, Springer Verlag, New York, 1977

[19] Schouten, J. A., *Ricci-Calculus. An introduction to Tensor Analysis and its Geometrical Applications*, Springer-Verlag, Berlin-Göttngen-Heidelberg, 1954.

[20] Shaikh, A. A. and Patra, A., *On a generalized class of recurrent manifolds*,
Archivum Mathematicum (BRNO), 46 (2010), 39--46.

[21] Shaikh, A. A. and Roy, I., *On quasi-generalized recurrent manifolds*, Mathematica Panonica, 21(2) (2010), 251--263 .

[22] Shaikh, A. A. and Roy, I., *On weakly generalized recurrent manifolds*, Ann. Univ. Sci. Budapest, Eötvös Sect. Math., 54 (2011) 35--45.

[23] Tamássy, L. and Binh, T. Q., *On weakly symmetric and weakly projective symmetric Riemannian manifolds*, Coll. Math. Soc., J. Bolyai, 56 (1989), 663--670.

[24] Yano, K. and Sawaki, S., *Riemannian manifolds admitting a conformal transformation group*, J. Diff. Geom., 2 (1968), 161--184.

[25] Walker, A. G., *On Ruse's space of recurrent curvature*, Proc. of London Math. Soc., 52 (1950), 36-54.