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Fixed Point of Mappings with Contractive iterates in a modular metric space

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Abstract: This paper is dealt with some existence theorems for fixed points of some contractive mappings over a modular metric space.Examples have been cited in strengthening of hypothesis of the theorems.

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1. INTRODUCTION

The concept of modular metric space have been initiated by Nakano([8]) and subsequently it was developed by Koshi and Shimogaki ([5]).Further the intensive study of the theory of modular metric space was done by Musielak et al. ([6]-([7]) and their collaborators.Now-a-days researchers are very much interested for searching of fixed points of contractive mappings over a modular metric space.In 2008, Chistyakov ([2]) introduced the notion of modular metric space generated by F-modular and obtained some results on such a space. Keeping on the same ideas Chistyakov ([3]) had been able to define the notion of a modular on an arbitrary set and developed the theory of metric spaces generated by metric modular called the modular metric spaces.Recently, there have been several generalization of contractive mappings possessing the fixed points in a setting of modular metric space by weakening the hypothesis,retaining the convergence property of the successive iterates to the unique fixed point.The contractive definition used in this paper are the generalization of contraction mapping due to Banach ([1]), Edelestein ([4]) and Seghal([9]) for metric spaces.In this paper several fixed point theorems and common fixed point theorem for contractive mapping are proved in a setting of modular metric space.

2. SOME BASIC IDEAS AND DEFINITIONS

In order to prove our main results, we firs recall the following definitions.

Definition 2.1.(see [3])Let X be a nonempty set.A function $\omega: (0,\infty)\times X\times X \to [0,\infty]$is said to be metric modular on$X$ if for all $x,y,z\in X$ the following conditions hold:

(i) $\omega_\lambda(x,y)=0$ for all $\lambda>0$ iff $x=y$

(ii) $\omega_\lambda(x,y)=\omega_\lambda(y,x)$ for all $\lambda>0$

(iii) $\omega_{\lambda+\mu}(x,y)\le\omega_\lambda(x,z)+\omega_\mu(z,y)$for all $\lambda,\mu>0$

The function $0<\lambda\to\omega_\lambda(x,y)\in[0,\infty]$ is non-increasing on $(0,\infty)$.If $0<\mu<\lambda$ then conditions (i)-(iii) imply that $\omega_\lambda(x,y)\le\omega_{\lambda-\mu}(x,x)+\omega_\mu(x,y)\le\omega_\mu(x,y)$.

Let us consider the following two sets (see[4])

$X_\omega\equiv X_\omega(x_0)=\{x\in X:\omega_\lambda(x,x_0)\to 0\ as\ \lambda\to\infty\}$ and

$X_\omega^*\equiv X_\omega^*(x_0)=\{x\in X:\exists\,\lambda=\lambda(x)>0\ s.t.\ \omega_\lambda(x,x_0)<\infty\}$

Definition 2.2.(see[2]) Let $X_\omega$be a modular metric space

(i) The sequence (x_n) in X_ω is said to be convergent to $x\in X_\omega$ if $\omega_\lambda(x_n,x)\to 0$ as $n\to\infty$ for all $\lambda>0$

(ii) The sequence (x_n) in X_ω is said to be cauchy if $\omega_\lambda(x_m,x_n)\to 0$ as $m,n\to\infty$ for all $\lambda>0$

Definition 2.3. (see [2]) A modular metric space $X_\omega$is said to be ω-complete if every cauchy sequence in $X_\omega$is convergent to an element in X_ω.

Theorem 2.4.(see[2]) *If ω is a metric modular on a set X, then the modular set X_ω is a metric space with metric given by $d_\omega^0(x,y)=inf\{\lambda>0:\omega_\lambda(x,y)<\lambda, x,y\in X_\omega\}$.*

Theorem 2.5.(see[2]) *Let ω be a modular on a set X, Given a sequence (x_n) in X_ω and $x\in X_\omega$ we have $d_\omega^0(x_n,x)\to 0$ as $n\to\infty$ iff $\omega_\lambda(x_n,x)\to 0$ as $n\to\infty$ for all $\lambda>0$.*

A similar assertion hold for cauchy sequences in X_ω.

3. MAIN RESULTS

Theorem 3.1. *Let X_ω be a ω-complete modular metric space and $T: X_\omega \to X_\omega$ be a map such that $\omega_\lambda(Tx,Ty) \le \phi(\omega_{2\lambda}(x,y))$,where $\phi: \mathbb{R} \to \mathbb{R}$ is upper semi continuous from the right and satisfy $\phi(t) < t$ for $t > 0$. Suppose that there exists an element $x = x(\lambda) \in X_\omega$ such that $\omega_\lambda(x,Tx) < \infty$, then T has a unique fixed point $x_0 \in X_\omega$ and for each $x \in X_\omega$, $T^n x \to x_0$ as $n \to \infty$.*

Proof. For each $x \in X_\omega$, let $x_n = T^n x$ and $d_n = \omega_{2\lambda}(x_n, x_{n+1})$. Therefore

$$\begin{aligned} d_n &= \omega_{2\lambda}(T^n x, T^{n+1}x) \\ &\le \omega_\lambda(T^n x, T^{n+1} x) \\ &= \omega_\lambda(T(T^{n-1}x), T(T^n x)) \\ &\le \phi\big(\omega_{2\lambda}(T^{n-1}x, T^n x)\big) \\ &< \omega_{2\lambda}(T^{n-1}x, T^n x) \\ &= d_{n-1} \end{aligned}$$

Therefore (d_n) is a decreasing sequence of real numbers which is bounded below.Thus (d_n) is convergent and $\lim d_n = d$ (say). Our claim is $d = 0$.Let $d > 0$, Since $d_n \le \phi(d_{n-1})$ and ϕ is upper semi continuous from right. we get $d \le \phi(d)$ which is a contradiction. Therefore we get $\lim \omega_\lambda(T^n x, T^{n+1}x) = 0$ i.e. $\lim \omega_\lambda(x_n, x_{n+1}) = 0$ for all $\lambda > 0$.For each $\varepsilon > 0$ and $\lambda > 0 \exists\, n_0 \in \mathbb{N}$ such that $\omega_{2\lambda}(x_n, x_{n+1}) < \varepsilon$, for all $n \in \mathbb{N}$ with $n \ge n_0$.Without loss of generality we assume $m > n$ and $m, n \in \mathbb{N}$. Since $\frac{\lambda}{2(m-n)} > 0$, therefore $\exists\, n_{\frac{\lambda}{2(m-n)}} \in \mathbb{N}$ such that $\omega_{\frac{\lambda}{m-n}}(x_n, x_{n+1}) < \frac{\varepsilon}{m-n}$ for all $n \ge n_{\frac{\lambda}{2(m-n)}}$. Now we have $m, n \ge n_{\frac{\lambda}{2(m-n)}}$,

$$\begin{aligned} \omega_\lambda(x_n, x_m) &\le \omega_{\frac{\lambda}{m-n}}(x_n, x_{n+1}) + \omega_{\frac{\lambda}{m-n}}(x_{n+1}, x_{n+2}) + \dots\dots\dots + \omega_{\frac{\lambda}{m-n}}(x_{m-1}, x_m) \\ &< \frac{\varepsilon}{m-n} + \frac{\varepsilon}{m-n} + \dots\dots\dots + \frac{\varepsilon}{m-n} = \varepsilon \end{aligned}$$

Therefore (x_n) is a cauchy sequence in X_ω. Since X_ω is complete,therefore (x_n) is converges to x_0 (say) in X_ω.Now we have $\omega_\lambda(x_{n+1}, Tx_0) = \omega_\lambda(Tx_n, Tx_0) \le \phi(\omega_{2\lambda}(x_n, x_0)) < \omega_{2\lambda}(x_n, x_0) \le \omega_\lambda(x_n, x_0)$. Therefore, $Tx_0 = \lim x_{n+1} = \lim x_n = x_0$ and $\lim T^n x = x_0$. For uniqueness, let $u(\ne x_0)$ be another fixed point of T, therefor we have, $\omega_\lambda(u, x_0) = \omega_\lambda(Tu, Tx_0) \le \phi(\omega_{2\lambda}(Tu, Tx_0)) < \omega_{2\lambda}(Tu, Tx_0) < \omega_{2\lambda}(u, x_0) < \omega_\lambda(u, x_0)$
which is a contradiction. Thus T has a unique fixed point in X_ω.

Example 3.2. Let $X = [0,1]$ and $\omega_\lambda(x,y) = \frac{|x-y|}{1+\lambda}$. Therefore $X_\omega = [0,1]$. Let $T: X_\omega \to X_\omega$ such that $Tx = \frac{x}{4}$ $\forall x \in X_\omega$ .Let $\phi: \mathbb{R} \to \mathbb{R}$ such that $\phi(t) = \frac{t}{2}$ if $t \ge 0$ and $\phi(t) = t - 1$ if $t < 0$. . ϕ is upper semi continuous from the right and satisfy $\phi(t) < t$ for $t > 0$.clearly $\omega_\lambda(Tx,Ty) \le \phi(\omega_{2\lambda}(x,y))$ $\forall x, y \in X_\omega$. Here 0 is the unique fixed point $T$and $T^n x \to 0$ as $n \to \infty$ for each $x \in X_\omega$.

Theorem 3.3. *Let X_ω be a ω-complete modular metric space and $T: X_\omega \to X_\omega$ be a map such that $\omega_\lambda(Tx,Ty) \le \phi(max\{\omega_{2\lambda}(x,y), \omega_{2\lambda}(x,Tx), \omega_{2\lambda}(y,Ty)\})$ for all $x, y \in X_\omega$ where $\phi: \mathbb{R} \to \mathbb{R}$ is upper semi continuous from the right and satisfy $\phi(t) < t$ for $t > 0$. Suppose that there exist an element $x = x(\lambda) \in X_\omega$ such that $\omega_\lambda(x,Tx) < \infty$, then T has a unique fixed point $y \in X_\omega$ and for each $x \in X_\omega T^n x \to y$ as $n \to \infty$.*

Proof: Since $x \in X_\omega$ such that $\omega_\lambda(x,Tx) < \infty$. Let $x_n = T^n x$ and $d_n = \omega_{2\lambda}(x_n, x_{n+1})$. Therefore

$$\begin{aligned} d_n &= \omega_{2\lambda}(T^n x, T^{n+1}x) \\ &\le \omega_\lambda(T(T^{n-1}x), T(T^n x)) \\ &= \omega_\lambda(T(Tx_{n-1}), T(Tx_n)) \\ &\le \phi(\max\{\omega_{2\lambda}(x_{n-1}, x_n), \omega_{2\lambda}(x_{n-1}, Tx_{n-1}), \omega_{2\lambda}(x_n, Tx_n)\}) \\ &\le \phi(\max\{\omega_{2\lambda}(x_{n-1}, x_n), \omega_{2\lambda}(x_{n-1}, x_n), \omega_{2\lambda}(x_n, x_{n+1})\}) \\ &\le \phi(\max\{\omega_{2\lambda}(x_{n-1}, x_n), \omega_{2\lambda}(x_n, x_{n+1})\}) \end{aligned}$$

If $\max\{\omega_{2\lambda}(x_{n-1}, x_n), \omega_{2\lambda}(x_n, x_{n+1})\} = \omega_{2\lambda}(x_n, x_{n+1}) = d_n$,then it is absurd. So $d_n \le \phi(d_{n-1}) < d_{n-1}$ i.e $d_n < d_{n-1}$ always. Hence by the Theorem 3.1 there exists a fixed point $y = \lim T^n x$, where $y \in X_\omega$
For uniqueness, let $v(\ne y)$ be another fixed point of T in X_ω, then we have

$$\omega_\lambda(y,v) = \omega_\lambda(Ty,Tv) \le \phi(\omega_\lambda(y,v)) < \omega_\lambda(y,v)$$

which is a contradiction. Thus T has a unique fixed point in X_ω.

Example 3.4. Let $X = [0,1]$ and $\omega_\lambda(x,y) = \frac{1}{1+\lambda} max\{|x|,|y|\}$ therefore $X_\omega = [0,1]$. Let $T: X_\omega \to X_\omega$ such that $Tx = \frac{x}{2}$. Consider $\phi: \mathbb{R} \to \mathbb{R}$ such that $\phi(t) = t$ if $t \geq 0$ and $\phi(t) = t - 1$ if $t < 0$. and satisfying $\omega_\lambda(Tx,Ty) \leq \phi(max\{\omega_{2\lambda}(x,y), \omega_{2\lambda}(x,Tx), \omega_{2\lambda}(y,Ty)\})$ for all $x, y \in X_\omega$. Here 0 is the unique fixed point T and $T^n x \to 0$ as $n \to \infty$ for each $x \in X_\omega$.

Theorem 3.5. *Let X_ω be a ω-complete modular metric space and (T_n) be a sequence of self mappings on X_ω such that $\omega_\lambda(T_n x, T_n y) \leq \phi(\max\{\omega_{2\lambda}(x,y), \omega_{2\lambda}(x,T_n x), \omega_{2\lambda}(y,T_n y)\})$ holds for all $x, y \in X_\omega$ where $\phi: \mathbb{R} \to \mathbb{R}$* is upper semi continuous from the right and satisfy $\phi(t) < t$ for $t > 0$ and for each n, (T_n) tends to point wise to a self map T on X_ω . Suppose that there exists an element $x = x(\lambda) \in X_\omega$ such that $\omega_\lambda(x,Tx) < \infty$, then T has a unique fixed point z in X_ω and $z_n \to z$ as $n \to \infty$ where z_n are the unique fixed points of each T_n.

Proof. For each $x, y \in X_\omega$ we have

$$\omega_\lambda(T_n x, T_n y) \leq \phi(\max\{\omega_{2\lambda}(x,y), \omega_{2\lambda}(x,T_n x), \omega_{2\lambda}(y,T_n y)\})$$

Taking limit on both side as $n \to \infty$ we get

$$\omega_\lambda(Tx,Ty) \leq \phi(\max\{\omega_{2\lambda}(x,y), \omega_{2\lambda}(x,Tx), \omega_{2\lambda}(y,Ty)\})$$

Hence T has a unique fixed point z(say) in X_ω .

Now $\omega_\lambda(z,z_n) = \omega_\lambda(Tz, T_n z_n) \leq \omega_{\frac{\lambda}{2}}(Tz, T_n z) + \omega_{\frac{\lambda}{2}}(T_n z, T_n z_n)$

$$\omega_{\frac{\lambda}{2}}(T_n z, T_n z_n) \leq \phi(\max\{\omega_\lambda(z,z_n), \omega_\lambda(z,T_n z), \omega_\lambda(z_n, T_n z_n)\}$$

$$= \phi(\max\{\omega_\lambda(z,z_n), \omega_\lambda(z,T_n z)\}$$

If $\max\{\omega_\lambda(z,z_n), \omega_\lambda(z,T_n z)\} = \omega_\lambda(z,z_n)$, then it is not possible.

So if $\max\{\omega_\lambda(z,z_n), \omega_\lambda(z,T_n z)\} = \omega_\lambda(z,T_n z) \leq \omega_{\frac{\lambda}{2}}(z,T_n z) = \omega_{\frac{\lambda}{2}}(Tz,T_n z)$, then we get $\omega_\lambda(z,z_n) \leq 2\omega_{\frac{\lambda}{2}}(Tz,T_n z)$. Since (T_n) converges pointwise to a function T, we have $z_n \to z$ as $n \to \infty$.

Example 3.6. Let $X = [0,1]$ and $\omega_\lambda(x,y) = \frac{1}{1+\lambda} max\{|x|,|y|\}$ therefore $X_\omega = [0,1]$. Let $T_n: X_\omega \to X_\omega$ be asequence of maps such that $T_n x = \frac{x}{n}$. Consider $\phi: \mathbb{R} \to \mathbb{R}$ such that $\phi(t) = t$ if $t \geq 0$ and $\phi(t) = t - 1$ if $t < 0$. Now $\omega_\lambda(T_n x, T_n y) \leq \phi(max\{\omega_{2\lambda}(x,y), \omega_{2\lambda}(x,T_n x), \omega_{2\lambda}(y,T_n y)\})$ all $x, y \in X_\omega$ and $\forall n \in \mathbb{N}$. Here $z_n = 0$ is the unique fixed point of each T_n for $n \geq 1$ and $T_n x \to Tx$ as $n \to \infty$ for each $x \in X_\omega$ where $T(x) = 0, \forall x \in X_\omega$ and

$z = 0$ is the unique fixed point of T in X_ω .

Theorem 3.7. *Let X_ω be a ω-complete modular metric space and F and G be two mapping of X into itself satisfying* $\omega_\lambda(Fx,Gy) \leq \phi(\max\{\omega_{2\lambda}(x,y), \omega_{2\lambda}(x,Fx), \omega_{2\lambda}(y,Gy)\})$ for all $x, y \in X_\omega$, where $\phi: \mathbb{R} \to \mathbb{R}$ is upper semi continuous from the right such that $\phi(t) < t$ for each $t > 0$. Then F and G have a unique common fixed point z in X_ω and $(FG)^n(x_1) \to z$ and $(GF)^n(x_0) \to z$ for some x_0, x_1 in X_ω .

Proof. Let $x_0 \in X_\omega$ and define (x_n) by $x_{2n+1} = Fx_{2n}$ and $x_{2n+2} = Gx_{2n+1}$.Now

$$\begin{aligned}\omega_\lambda(x_{2n+1}, x_{2n+2}) &= \omega_\lambda(Fx_{2n}, Gx_{2n+1}) \\ &\leq \phi(\max\{\omega_{2\lambda}(x_{2n}, x_{2n+1}), \omega_{2\lambda}(x_{2n}, Fx_{2n}), \omega_{2\lambda}(x_{2n+1}, Gx_{2n+1})\}) \\ &= \phi(\max\{\omega_{2\lambda}(x_{2n}, x_{2n+1}), \omega_{2\lambda}(x_{2n}, x_{2n+1}), \omega_{2\lambda}(x_{2n+1}, x_{2n+2})\})\end{aligned}$$

$= \phi(\max\{\omega_{2\lambda}(x_{2n}, x_{2n+1}), \omega_{2\lambda}(x_{2n+1}, x_{2n+2})\}) < \omega_{2\lambda}(x_{2n}, x_{2n+1}) \leq \omega_\lambda(x_{2n}, x_{2n+1})$

Similarly $\omega_\lambda(x_{2n}, x_{2n+1}) < \omega_\lambda(x_{2n-1}, x_{2n})$ Therefore we get $\omega_\lambda(x_n, x_{n+1}) < \omega_\lambda(x_{n-1}, x_n)$ for all n. So, $(x_n)$is a cauchy sequence in X_ω. Let $\lim x_n = z \in X_\omega$. Our claim $Fz = z$. If $Fz \neq z$ then

$$\omega_{2\lambda}(Fz,z) \leq \omega_\lambda(Fz, x_{2n+2}) + \omega_\lambda(x_{2n+2}, z)$$

It is easy to see that $\omega_{2\lambda}(Fz,z) = 0$ implies that $Fz = z$. Similarly we see that $Gz = z$. If possible let $w \in X_\omega$ be another common fixed pont of F and G.Now,

$$\omega_\lambda(z,w) = \omega_\lambda(Fz,Gw) = \phi(\max\{\omega_{2\lambda}(z,w), \omega_{2\lambda}(z,Fz), \omega_{2\lambda}(w,Gw)\}) = \phi(\omega_{2\lambda}(z,w)) < \omega_{2\lambda}(z,w) < \omega_\lambda(z,w)$$

which is a contradiction .

Also by routine verification it can be easily checked that $(FG)^n(x_1) \to z$ and $(GF)^n(x_0) \to z$.

Example 3.8. Let $X = [0,1]$ *and* $\omega_\lambda(x,y) = \frac{1}{1+\lambda} max\{|x|,|y|\}$. Therefore $X_\omega = [0,1]$. Let $T_n: X_\omega \to X_\omega$ such that $Fx = \frac{x}{2}$ and $Gx = \frac{x}{4}$ .Consider $\phi: \mathbb{R} \to \mathbb{R}$ such that $\phi(t) = t$ if $t \geq 0$ and $\phi(t) = t - 1$ if $t < 0$.

Then $\omega_\lambda(Fx, Gy) \le \phi(max\{\omega_{2\lambda}(x,y), \omega_{2\lambda}(x,Fx), \omega_{2\lambda}(y,Gy)\})$for all $x, y \in X_\omega$. Here F and G have a unique common fixed point 0 in X_ω.

Theorem 3.9. *Let X_ω be a ω-complete modular metric space and (T_n) be a sequence of mappings of X into itself.Suppose there exists a sequence of non-negative integer(m_n) such that for all $x, y \in X_\omega$ and for every $i, j(i \neq j)$ satisfying*

$$\omega_\lambda(T_i^{m_i}x, T_i^{m_i}y) \le \phi(\max\{\omega_{2\lambda}(x,y), \omega_{2\lambda}(x, T_i^{m_i}x), \omega_{2\lambda}(y, T_i^{m_i}y)\})$$

for all $x, y \in X_\omega$where $\phi: \mathbb{R} \to \mathbb{R}$ is upper semi continuous from the right and satisfy $\phi(t) < t$ for $t > 0$. Then the mapping (T_n) have a unique common fixed point in X_ω .

Proof. Define $G_i = T_i^{m_i}$, $i = 1,2,\ldots$

$$\omega_\lambda(G_i x, G_j y) \le \phi(\max\{\omega_{2\lambda}(x,y), \omega_{2\lambda}(x, G_i x), \omega_{2\lambda}(y, G_j y)\})$$

Let $x_0 \in X_\omega$, Define $x_n = G_n(x_{n-1})$

$$\begin{aligned}\omega_\lambda(x_n, x_{n+1}) &= \omega_\lambda(G_n(x_{n-1}), G_{n+1}(x_n))\\ &\le \phi(\max\{\omega_{2\lambda}(x_{n-1}, x_n), \omega_{2\lambda}(x_{n-1}, G_n x_{n-1}), \omega_{2\lambda}(x_n, G_{n+1}x_n)\})\\ &\le \phi(\max\{\omega_{2\lambda}(x_{n-1}, x_n), \omega_{2\lambda}(x_{n-1}, x_n), \omega_{2\lambda}(x_n, x_{n+1})\})\\ &\le \phi\{\omega_{2\lambda}(x_{n-1}, x_n)\})\\ &< \omega_{2\lambda}(x_{n-1}, x_n) < \omega_{2\lambda}(x_{n-1}, x_n)\end{aligned}$$

It is easy to verify that (x_n) is a cauchy sequence in X_ω and converges to $z \in X_\omega$ (say) .

Now, $\omega_\lambda(G_n z, G_{m+1}x_m) \le \phi(\max\{\omega_{2\lambda}(z, x_m), \omega_{2\lambda}(z, G_n z), \omega_{2\lambda}(x_m, G_{m+1}(x_m))\}$

$$\le \phi(\max\{\omega_{2\lambda}(z, x_m), \omega_{2\lambda}(z, G_n z), \omega_{2\lambda}(x_m, x_{m+1})\}$$

But $\max\{\omega_{2\lambda}(z, x_m), \omega_{2\lambda}(z, G_n z), \omega_{2\lambda}(x_m, x_{m+1})\} \neq \omega_{2\lambda}(z, G_n z)$. Otherwise we arrive at a contradiction.Therefore, $\omega_{2\lambda}(z, G_n z) = 0$, that is, $z = G_n z$ for all $n \in \mathbb{N}$. Uniqueness is also clear. For each $n \in \mathbb{N}$, $T_n z = T_n(G_n z) = T_n(T_n^{m_n} z) = T_n^{m_n}(T_n z)$ which implies $T_n z$ is a fixed point of G_n. By uniqueness we get $T_n z = z$ and z is a common fixed point of T_n.Uniqueness also readily follows.

REFERENCES

1. Banach,S. *Surles operations dens les ensembles abstraits et leur application dux equations untegrales*,Fund. Math. **3**,133-181(1922).
2. Chistyakov .W, *Modular metric space,I:Basic Concepts*,Nonlinear Anal.**72**(1),1-14(2010).
3. Chistyakov .W, *Modular metric space,II:Applicatuin to superposition operators*,Nonlinear Anal.**72**(1),15-30(2010).
4. Edelestein.M, *An extention of Banach contraction principle*,Proc. Amer. Math.Soc.,17-10(1961).
5. Koshi,Shimogaki, *On F-form of quasi- modular space*,J Fac Sci Hokkaido Univ Ser I.**15**,(3-4):202-218(1961).
6. Musielak.J and Orlicz.W *On modular spaces. Studia Math.***18** 49-65(1959).
7. Musielak.J and Orlicz.W *Some remark on modular spaces. Bull Acad Polon Sci Sr Sci Math Astron Phys.***7** 661-668(1959).
8. Nakano.H, *Modulared semi Ordered Linear spaces*,In Tokyo Math Book Ser, Vol.1 Maruzen Co, Tokyo (1950).
9. Seghal.V.M, *A fixed point theorem for mapping with a contractive iterate*,Proc.Amer.Maths. Soc.**23**,631-644(1969).
10. Turpin,Ph: *Fubini inequalities and bounded multiplier property in generalised modular space. Comment. Math Tomus specialis in honorem Ladislai Orlicz I.* 331-353(1978).