Research Guru
Online Journal of Multidisciplinary Subjects (ISSN : 2349-266X)
UGC Approved Journal No. 63726
Impact Factor:4.081
Volume-12, Issue-4, March-2019 www.researchguru.net

Certain Properties of q-Fractional Integral Operators Associated with q-Analogue of Generalized M-Series

Shakir Hussain Malik[1], Renu Jain[2] and Javid Majid[3]

malikshakir774@gmail.com, renujain3@rediffmail.com, javidmajidmaths@gmail.com

School of Mathematics and Allied Sciences Jiwaji University, Gwalior, (M.P.) India.

Abstract: In this paper, we introduce the q-analogue of generalized M-series ${}_p^{\alpha}M_r^{\beta}(z;q)$, its special cases and evaluate certain q-fractional integrals and derivatives involving q-generalized M-series. Some special cases have also been discussed.

Keywords: M-series, q-beta function, left and right sided Riemann-Liouville q-derivative, generalized q-Mittag-Leffler function.

1. Introduction

In 2008, the M-series studied by Sharma [4] and is given by:

$$ {}_p^{\alpha}M_r\left(a_1,\dots\dots,a_p;b_1,\dots\dots,b_r;z\right) = {}_p^{\alpha}M_r(z)$$

$$ {}_p^{\alpha}M_r(z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots\dots (a_p)_k}{(b_1)_k \dots\dots (b_r)_k} \frac{z^k}{\Gamma(\alpha k+1)} \qquad (1.1)$$

where, $z,\alpha \in C$, $R(\alpha) > 0$, and $(a_j)_k$, $(b_j)_k$ are the pochammer symbols.

The series in (1.1) is defined when none of the parameters $b_j;s;j = 1,2,\dots\dots,r,$ is a negative integer or zero, If any numerator parameter a_j is a negative integer or zero, then the series terminates to a polynomial in z. From the ratio test it is evident that the series in (1.1) is convergent for all z if $p \le r$, also if $p = r+1$ it is convergent absolutely or conditionally when $|z| = 1$, and divergent if $p > r+1$.

Further extension of both Mittag–Leffler function and generalized hypergeometric function ${}_pF_q$ is called generalized M-series introduced and studied by Sharma and Jain [5]:

$$ {}_p^{\alpha}M_r^{\beta}\left(a_1,a_2,\dots,a_p;b_1,b_2,\dots b_r;z\right) = {}_p^{\alpha}M_r^{\beta}\left((a_j)_1^p;(b_j)_1^r;z\right)$$

$$ {}_p^{\alpha}M_r^{\beta}(z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_r)_k} \frac{z^k}{\Gamma(\alpha k+\beta)}; z,\alpha,\beta \in \mathbb{C} \qquad (1.2)$$

The series in (1.2) is convergent for all z if $p \le r + Re(\alpha)$, also it is convergent for $|z| < \delta = \alpha^{\alpha}$ if $p = r + Re(\alpha)$ and divergent if $p > r + Re(\alpha)$.

2. MATHEMATICAL PRELIMINARIES

In the theory of q-series by Gasper and Rahman [1], for real or complex a and $|q| < 1$,

The q-shifted factorial (q-analogue of Pochhammer symbol) is defined as

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), n \in N \qquad (2.1)$$

with $(a;q)_0 = 1$, $q \neq 1$.

If we consider $(a;q)_\infty$ then as the infinite product diverges when $a \neq 0$ and $|q| \geq 1$, therefore whenever $(a;q)_\infty$ appears in a formula, we shall assume that $|q| < 1$.
Also, for any complex number α, we have

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty}, \tag{2.2}$$

where the principal value of q^α is taken.
The q-analogue of power function $(z-a)^\alpha$ is defined as

$$(z-a)^0 = 1, \quad (z-a)_q^\alpha = z^\alpha (a/z;q)_\alpha$$

$$= z^\alpha \prod_{j=0}^{\infty} \left[\frac{1-(a/z)q^j}{1-(a/z)q^{j+\alpha}}\right] = z^\alpha \frac{(a/z;q)_\infty}{(q^\alpha a/z;q)_\infty}, 0 < |q| < 1, (z \neq 0). \tag{2.3}$$

Also, Predrag M. Rajkovic, et al. [9], defined a q-derivative of a function $f(z)$ by

$$\left(D_q f\right)(z) = \frac{f(z) - f(qz)}{(1-q)z}, (z \neq 0, q \neq 1). \tag{2.4}$$

and

$$\lim_{q \to 1} D_q f(z) = \frac{df(z)}{dz} \tag{2.5}$$

$$(D_{\frac{1}{q}}^n f)\,(x) = q^n (D_q^n f)\left(\frac{x}{q^n}\right) \tag{2.6}$$

$$D_q(f(x)g(x)) = g(x)D_q f(x) + f(qx)D_q g(x) \tag{2.7}$$

$$D_q\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)D_q f(x) - f(x)D_q g(x)}{g(x)g(qx)} \tag{2.8}$$

$$D_q^n z^\mu = \frac{\Gamma_q(\mu+1)}{\Gamma_q(\mu-n+1)} z^{\mu-n}, Re(\mu) + 1 > 0. \tag{2.9}$$

Further, the $\Gamma_q(z)$ satisfies the functional equation,

$$\Gamma_q(z+1) = \frac{1-q^z}{1-q}\Gamma_q(z) \tag{2.10}$$

Again, the q-analogue of the beta function is defined by

$$B_q(x,y) = \int_0^1 t^{x-1} \frac{(tq;q)_\infty}{(tq^y;q)_\infty} d_q(t) \tag{2.11}$$

The relation between q-beta function and q-gamma function is

$$B_q(x,y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}, (Re(x) > 0, Re(y) > 0) \tag{2.12}$$

Definition 1:- Let $0 < \alpha \leq 1$. The left-sided and right-sided Riemann-Liouville q-fractional operator [2], [3] are given by the formulas

$$I_{q,a+}^{\alpha} f(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_a^x ({}^{qt}/_{x}\,; q)_{\alpha-1}\, f(t) d_q t \qquad (2.13)$$

and

$$I_{q,b-}^{\alpha} f(x) = \frac{1}{\Gamma_q(\alpha)} \int_{qx}^b t^{\alpha-1} ({}^{qx}/_{t}\,; q)_{\alpha-1} f(t) d_q t \qquad (2.14)$$

Definition 2:- Let $\alpha > 0$ and $[\alpha] = m$. The left and right-side Riemann-Liouville fractional q-derivatives of order α [2], [3] are defined by

$$D_{q,a+}^{\alpha} f(x) := D_q^m\, I_{q,a+}^{m-\alpha} f(x) \qquad (2.15)$$

and

$$D_{q,b-}^{\alpha} f(x) := \left(\frac{-1}{q}\right)^m D_{q^{-1}}^m\, I_{q,b-}^{m-\alpha} f(x) \qquad (2.16)$$

3. Main results

In this section, we define the q-analogue of generalized M-series:

Definition 3:- For $\alpha, \beta \in C, R(\alpha) > 0$ and $|q| < 1$ the function ${}_p^{\alpha}M_r^{\beta}(z; q)$ is defined as

$${}_p^{\alpha}M_r^{\beta}(z; q) = {}_p^{\alpha}M_r^{\beta}(a_1, a_2, ... a_p; b_1, b_2, ... b_r; z; q)$$

$$= \sum_{k=0}^{\infty} \frac{(a_1; q)_k \ldots\ldots (a_p; q)_k}{(b_1; q)_k \ldots\ldots (b_r; q)_k\, (q; q)_k} \frac{z^k}{\Gamma_q(\alpha k + \beta)} \qquad (3.1)$$

Where $(a_j; q)_k, (b_j; q)_k$ are the q-analogue of Pochhammer symbol and $\Gamma_q(\lambda)$ is the q-gamma function.

Moreover, in view of the relation

$${}_p^{\alpha}M_r^{\beta}(z; q) = \sum_{k=0}^{\infty} \frac{(a_1; q)_k \ldots\ldots (a_p; q)_k}{(b_1; q)_k \ldots\ldots (b_r; q)_k\, (q; q)_k} \frac{z^k}{\Gamma_q(\alpha k + \beta)}$$

$$= \frac{\prod_{j=1}^{r} \Gamma_q b_j}{\prod_{j=1}^{p} \Gamma_q a_j}\, {}_{p+1}\psi_{r+1}\left[\begin{matrix} (a_i, 1)_1^p, (1,1) \\ (b_j, 1)_1^r,\ \ (\beta, \alpha) \end{matrix}; z; q\right]$$

$$= \frac{\prod_{j=1}^{r} \Gamma_q b_j}{\prod_{j=1}^{p} \Gamma_q a_j} H_{p+1,r+2}^{1,p+1}\left[-z; q \left| \begin{matrix} (1-a_i, 1)_1^p, (0,1) \\ (0,1), (1-b_j, 1)_1^r, (1-\beta, \alpha) \end{matrix} \right.\right]$$

The function ${}_p^{\alpha}M_r^{\beta}(z; q)$ converges under the convergence conditions of the well-known Fox-Wright generalized hypergeometric function and generalized H-function which are as follows, the integral converges if $Re[s \log(z) - \log sin\pi s] < 0$, on the contour C, where $0 < |q| < 1, \log q = -\omega = (\omega_1 + i\omega_2)$. ω_1 *and* ω_2 being real, verified by Saxena, et. al.[10].

Some special cases of the ${}_p^{\alpha}M_r^{\beta}(z; q)$-function are the following:

(i) The q-Mittag-Leffler function [6]: when there is no upper and lower parameters $(p = r = 0)$, we have

$$E_{\alpha,\beta}(z;q) = {}^{\alpha}_{0}M^{\beta}_{0}(-;-;z;q) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma_q(\alpha k+\beta)} \quad (3.2)$$

(ii) The generalized q-Mittag-Leffler function introduced by S.K.Sharma and R.Jain [7], is obtained from (3.1) for $p = r = 1;\ a = \gamma \in C;\ b = 1$:

$$E^{\gamma}_{\alpha,\beta}(z;q) = \sum_{k=0}^{\infty} \frac{(q^{\gamma};q)_k}{(q;q)_k} \frac{z^k}{\Gamma_q(\alpha k+\beta)} = {}^{\alpha}_{1}M^{\beta}_{1}(q^{\gamma};q;z) \quad (3.3)$$

(iii) The q-generalized M-series can be represented as a special case of the q-Wright generalized hypergeometric function

$$ {}^{\alpha}_{p}M^{\beta}_{r}((a_j;q)_1^p;(b_j;q)_1^r;z;q) = \frac{\prod_{j=1}^{r}\Gamma_q(b_j)}{\prod_{j=1}^{p}\Gamma_q(a_j)} \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p}\Gamma_q(a_i+k)}{\prod_{j=1}^{r}\Gamma_q(b_j+k)} \frac{z^k}{\Gamma_q(\alpha k+\beta)} \quad (3.4)$$

4. q-fractional calculus of q-generalized M-series:

Theorem 1:- Let $\alpha > 0, \beta > 0, \gamma > 0, a \in R$ and $I^{\alpha}_{q,0+}$ be the left-sided operator of Riemann-Liouville q-fractional integral (2.13). Then there holds the formula

$$\left(I^{\alpha}_{q,0+}\left[t^{\gamma-1}\ {}^{\beta}_{p}M^{\gamma}_{r}((a_j;q)_1^p;(b_j;q)_1^r;at^{\beta};q)\right]\right)(x) = x^{\alpha+\gamma-1}\ {}^{\beta}_{p}M^{\alpha+\gamma}_{r}((a_j;q)_1^p;(b_j;q)_1^r;ax^{\beta};q) \quad (4.1)$$

Proof: By using definition of q-generalized M-series (3.1) and q-fractional integral formula (2.13), we obtained

$$K \equiv \left(I^{\alpha}_{q,0+}\left[t^{\gamma-1}\ {}^{\beta}_{p}M^{\gamma}_{r}((a_j;q)_1^p;(b_j;q)_1^r;at^{\beta};q)\right]\right)(x)$$

$$= \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x ({}^{qt}/_{x};q)_{\alpha-1}\, t^{\gamma-1} \sum_{k=0}^{\infty} \frac{(a_1;q)_k \ldots.(a_p;q)_k}{(b_1;q)_k \ldots.(b_r;q)_k\,(q;q)_k} \frac{a^k t^{\beta k}}{\Gamma_q(\beta k+\gamma)} d_q t$$

$$= \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \sum_{k=0}^{\infty} \frac{(a_1;q)_k \ldots.(a_p;q)_k}{(b_1;q)_k \ldots.(b_r;q)_k\,(q;q)_k} \frac{a^k}{\Gamma_q(\beta k+\gamma)} \int_0^x ({}^{qt}/_{x};q)_{\alpha-1} t^{\beta k+\gamma-1} d_q t$$

Now, using equation (2.3) above expression reduce to

$$K = \frac{1}{\Gamma_q(\alpha)} \sum_{k=0}^{\infty} \frac{(a_1;q)_k \ldots.(a_p;q)_k}{(b_1;q)_k \ldots.(b_r;q)_k\,(q;q)_k} \frac{a^k}{\Gamma_q(\beta k+\gamma)} \times \int_0^x x^{\alpha-1} \frac{\left(\frac{qt}{x};q\right)_{\infty}}{\left(\frac{q^{\alpha}t}{x};q\right)_{\infty}} t^{\beta k+\gamma-1} d_q t$$

$$= \frac{1}{\Gamma_q(\alpha)} \sum_{k=0}^{\infty} \frac{(a_1;q)_k \ldots.(a_p;q)_k}{(b_1;q)_k \ldots.(b_r;q)_k\,(q;q)_k} \frac{a^k}{\Gamma_q(\beta k+\gamma)} x^{\alpha-1} \int_0^x \frac{\left(\frac{qt}{x};q\right)_{\infty}}{\left(\frac{q^{\alpha}t}{x};q\right)_{\infty}} t^{\beta k+\gamma-1} d_q t$$

Substituting $t = \xi x \quad \Rightarrow \; d_q t = x d_q \xi$, we get

$$K = \frac{1}{\Gamma_q(\alpha)} \sum_{k=0}^{\infty} \frac{(a_1;q)_k \ldots\ldots (a_p;q)_k}{(b_1;q)_k \ldots\ldots (b_r;q)_k \; (q;q)_k} \frac{a^k}{\Gamma_q(\beta k+\gamma)} \; x^{\alpha+\beta k+\gamma-1}$$

$$\times \int_0^1 (\xi)^{\beta k+\gamma-1} \frac{(\xi q;q)_\infty}{(\xi q^\alpha;q)_\infty} \, d_q\xi$$

Using equation (2.11), it takes the form

$$K = \frac{1}{\Gamma_q(\alpha)} \sum_{k=0}^{\infty} \frac{(a_1;q)_k \ldots\ldots (a_p;q)_k}{(b_1;q)_k \ldots\ldots (b_r;q)_k \; (q;q)_k} \frac{a^k}{\Gamma_q(\beta k+\gamma)} \; x^{\alpha+\beta k+\gamma-1} \quad B_q(\beta k+\gamma, \alpha)$$

Also, using equation (2.12) and on simplification, the RHS of above equation reduce to

$$K = x^{\alpha+\gamma-1} \sum_{k=0}^{\infty} \frac{(a_1;q)_k \ldots\ldots (a_p;q)_k}{(b_1;q)_k \ldots\ldots (b_r;q)_k \; (q;q)_k} \frac{(a x^\beta)^k}{\Gamma_q(\beta k+\gamma+\alpha)}$$

$$= \; x^{\alpha+\gamma-1} \, {}_p^\beta M_r^{\alpha+\gamma}((a_j;q)_1^p; (b_j;q)_1^r; a x^\beta; q)$$

This completes the proof of (4.1).

Remark 1:- If we put $p = r = 1, \; a = \delta \in C, \; b = 1$ in (4.1), we shall obtain

$$\left(I_{q,0+}^\alpha \left[t^{\gamma-1} \, E_{\beta,\gamma}^\delta \left(a t^\beta; q\right)\right]\right)(x) = \; x^{\alpha+\gamma-1} \, E_{\beta,\alpha+\gamma}^\delta \left(a x^\beta; q\right)$$

Theorem 2:- Let $\alpha > 0, \beta > 0, \gamma > 0, a \in R$ and $I_{q,-}^\alpha$ be the right-sided operator of Riemann- Liouville q-fractional integral (2.14). Then there holds the formula

$$\left(I_{q,-}^\alpha \left[t^{-\alpha-\gamma} \, {}_p^\beta M_r^\gamma((a_j;q)_1^p; (b_j;q)_1^r; a t^{-\beta}; q)\right]\right)(x)$$

$$= \frac{x^{-\gamma}}{q} \, {}_p^\beta M_r^{\alpha+\gamma}((a_j;q)_1^p; \left(b_j;q\right)_1^r; a x^{-\beta}; q) \qquad (4.2)$$

Proof: By using definition of q-generalized M-series (3.1) and q-fractional integral formula (2.14), we obtained

$$K \equiv \left(I_{q,-}^\alpha \left[t^{-\alpha-\gamma} \, {}_p^\beta M_r^\gamma((a_j;q)_1^p; (b_j;q)_1^r; a t^{-\beta}; q)\right]\right)(x)$$

$$= \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_{qx}^{\infty} ({}^{qx}/_t; q)_{\alpha-1} \, t^{-\alpha-\gamma} \sum_{k=0}^{\infty} \frac{(a_1;q)_k \ldots\ldots (a_p;q)_k}{(b_1;q)_k \ldots\ldots (b_r;q)_k \; (q;q)_k} \frac{a^k t^{-\beta k}}{\Gamma_q(\beta k+\gamma)} d_q t$$

$$= \frac{1}{\Gamma_q(\alpha)} \sum_{k=0}^{\infty} \frac{(a_1;q)_k \ldots\ldots (a_p;q)_k}{(b_1;q)_k \ldots\ldots (b_r;q)_k \; (q;q)_k} \frac{a^k}{\Gamma_q(\beta k+\gamma)} \int_{qx}^{\infty} ({}^{qx}/_t; q)_{\alpha-1} \, t^{-\gamma-\beta k-1} d_q t$$

Using equation (2.3), we get

$$K = \frac{1}{\Gamma_q(\alpha)} \sum_{k=0}^{\infty} \frac{(a_1;q)_k \dots (a_p;q)_k}{(b_1;q)_k \dots (b_r;q)_k \; (q;q)_k} \frac{a^k}{\Gamma_q(\beta k+\gamma)} \times \int_{qx}^{\infty} t^{-\gamma-\beta k-1} \frac{\left(\frac{qx}{t};q\right)_\infty}{\left(\frac{q^{\alpha-1}qx}{t};q\right)_\infty} d_q t$$

Substituting $t = \frac{qx}{\xi} \quad \Rightarrow d_q t = \frac{-qx}{\xi(\xi q)} d_q \xi = \frac{-x}{\xi^2} d_q \xi$, we get

$$K = \frac{1}{\Gamma_q(\alpha)} \sum_{k=0}^{\infty} \frac{(a_1;q)_k \dots (a_p;q)_k}{(b_1;q)_k \dots (b_r;q)_k \; (q;q)_k} \frac{a^k}{\Gamma_q(\beta k+\gamma)} x^{-\gamma-\beta k} q^{-\gamma-\beta k-1} \times \int_0^1 \xi^{\gamma+\beta k-1} \frac{(\xi;q)_\infty}{(q^{\alpha-1}\xi;q)_\infty} d_q \xi$$

Replacing (ξ) by (μq), we get

$$K = \frac{1}{\Gamma_q(\alpha)} \sum_{k=0}^{\infty} \frac{(a_1;q)_k \dots (a_p;q)_k}{(b_1;q)_k \dots (b_r;q)_k \; (q;q)_k} \frac{a^k}{\Gamma_q(\beta k+\gamma)} \times x^{-\gamma-\beta k} q^{-1} \int_0^1 (\mu)^{\gamma+\beta k-1} \frac{(\mu q;q)_\infty}{(\mu q^{\alpha};q)_\infty} d_q(\mu)$$

Using equation (2.11), it takes the form

$$K = \frac{1}{\Gamma_q(\alpha)} \sum_{k=0}^{\infty} \frac{(a_1;q)_k \dots (a_p;q)_k}{(b_1;q)_k \dots (b_r;q)_k \; (q;q)_k} \frac{a^k}{\Gamma_q(\beta k+\gamma)} \frac{x^{-\gamma-\beta k}}{q} B_q(\gamma+\beta k, \alpha)$$

Also, using the relation (2.12) and on simplification, the RHS of above equation reduce to

$$K = \frac{x^{-\gamma}}{q} \sum_{k=0}^{\infty} \frac{(a_1;q)_k \dots (a_p;q)_k}{(b_1;q)_k \dots (b_r;q)_k \; (q;q)_k} \frac{\left(ax^{-\beta}\right)^k}{\Gamma_q(\beta k+\gamma+\alpha)}$$

$$= \frac{x^{-\gamma}}{q} \; {}_p^{\beta}M_r^{\alpha+\gamma}((a_j;q)_1^p; (b_j;q)_1^r; ax^{-\beta}; q)$$

This completes the proof of (4.2).

Remark 2:- If we put $p = r = 1, a = \delta \in C, b = 1$ in (4.2), we shall obtain

$$\left(I_{q,-}^{\alpha}\left[t^{-\alpha-\gamma} E_{\beta,\gamma}^{\delta}\left(at^{-\beta};q\right)\right]\right)(x) = \frac{x^{-\gamma}}{q} E_{\beta,\alpha+\gamma}^{\delta}(ax^{-\beta};q)$$

Theorem 3:- Let $\alpha > 0, \beta > 0, \gamma > 0, a \in R$ and $D_{q,0+}^{\alpha}$ be the left-sided operator of Riemann-Liouville q-fractional derivative (2.15). Then there holds the formula

$$\left(D_{q,0+}^{\alpha}\left[t^{\gamma-1} \; {}_p^{\beta}M_r^{\gamma}((a_j;q)_1^p; (b_j;q)_1^r; at^{\beta}; q)\right]\right)(x) = x^{\gamma-\alpha-1} \; {}_p^{\beta}M_r^{\gamma-\alpha}((a_j;q)_1^p; \left(b_j;q\right)_1^r; ax^{\beta}; q) \quad (4.3)$$

Proof: Using (3.1) and q-fractional integral formula (2.15), we obtained

$$K \equiv \left(D_{q,0+}^{\alpha}\left[t^{\gamma-1}\ {}_{p}^{\beta}M_{r}^{\gamma}\left((a_j;q)_1^p;(b_j;q)_1^r;at^{\beta};q\right)\right]\right)(x)$$

$$= D_q^k\left(I_{q,0+}^{k-\alpha}\left[t^{\gamma-1}\ {}_{p}^{\beta}M_{r}^{\gamma}\left((a_j;q)_1^p;(b_j;q)_1^r;at^{\beta};q\right)\right]\right)(x)$$

$$= \frac{1}{\Gamma_q(k-\alpha)}\, D_q^k \int_0^x x^{k-\alpha-1}\left({}^{qt}/_{x};q\right)_{k-\alpha-1}\sum_{k=0}^{\infty}\frac{(a_1;q)_k \ldots\ldots (a_p;q)_k}{(b_1;q)_k \ldots\ldots (b_r;q)_k\,(q;q)_k}\,\frac{a^k t^{\beta k+\gamma-1}}{\Gamma_q(\beta k+\gamma)}\, d_q t$$

Using equation (2.3), we get

$$K = \frac{1}{\Gamma_q(k-\alpha)}\sum_{k=0}^{\infty}\frac{(a_1;q)_k \ldots\ldots (a_p;q)_k}{(b_1;q)_k \ldots\ldots (b_r;q)_k\,(q;q)_k}\,\frac{a^k}{\Gamma_q(\beta k+\gamma)}$$
$$\times D_q^k\, x^{k-\alpha-1}\int_0^x t^{\gamma+\beta k-1}\,\frac{\left(\frac{qt}{x};q\right)_\infty}{\left(\frac{q^{k-\alpha}t}{x};q\right)_\infty}\, d_q t$$

Substituting $t = \xi x \;\Rightarrow\; d_q t = x d_q \xi$, we get

$$K = \frac{1}{\Gamma_q(k-\alpha)}\sum_{k=0}^{\infty}\frac{(a_1;q)_k \ldots\ldots (a_p;q)_k}{(b_1;q)_k \ldots\ldots (b_r;q)_k\,(q;q)_k}\,\frac{a^k}{\Gamma_q(\beta k+\gamma)}$$
$$\times D_q^k\, x^{k-\alpha+\beta k+\gamma-1}\int_0^1 \xi^{\gamma+\beta k-1}\,\frac{(\xi q;q)_\infty}{(\xi q^{k-\alpha};q)_\infty}\, d_q \xi$$

Using equation (2.11), it takes the form

$$K = \frac{1}{\Gamma_q(k-\alpha)}\sum_{k=0}^{\infty}\frac{(a_1;q)_k \ldots\ldots (a_p;q)_k}{(b_1;q)_k \ldots\ldots (b_r;q)_k\,(q;q)_k}\,\frac{a^k}{\Gamma_q(\beta k+\gamma)}$$
$$\times D_q^k\, x^{k-\alpha+\beta k+\gamma-1}\, B_q(\beta k+\gamma, k-\alpha)$$

Also, using the relation (2.12) and on simplification, the RHS of above equation reduce to

$$K = x^{\gamma-\alpha-1}\sum_{k=0}^{\infty}\frac{(a_1;q)_k \ldots\ldots (a_p;q)_k}{(b_1;q)_k \ldots\ldots (b_r;q)_k\,(q;q)_k}\,\frac{(ax^{\beta})^k}{\Gamma_q(\beta k+\gamma-\alpha)}$$

$$= x^{\gamma-\alpha-1}\ {}_{p}^{\beta}M_{r}^{\gamma-\alpha}\left((a_j;q)_1^p;(b_j;q)_1^r;ax^{\beta};q\right)$$

This completes the proof of (4.3).

Remark 3:- If we put $p = r = 1,\ a = \delta \in C,\ b = 1$ in (4.3), we shall obtain

$$\left(D_{q,0+}^{\alpha}\left[t^{\gamma-1}\, E_{\beta,\gamma}^{\delta}\left(at^{\beta};q\right)\right]\right)(x) = x^{\gamma-\alpha-1}E_{\beta,\gamma-\alpha}^{\delta}(ax^{\beta};q)$$

Theorem 4:- Let $\alpha > 0,\ \beta > 0,\ \gamma > 0,$ with $\gamma - \alpha + \{\alpha\} > 1$ and $\alpha \in R$ and let $D_{q,-}^{\alpha}$ be the right-sided operator of Riemann-Liouville q-fractional derivative (2.16). Then there holds the formula

$$\left(D_{q,-}^{\alpha}\left[\, t^{\alpha-\gamma}\ \ {}_{p}^{\beta}M_{r}^{\gamma}\,((a_j;q)_1^p;(b_j;q)_1^r;at^{-\beta};q)\right]\right)(x)$$
$$= \frac{1}{q^{k+1}}x^{-\gamma}\ {}_{p}^{\beta}M_{r}^{\gamma-\alpha}((a_j;q)_1^p;\left(b_j;q\right)_1^r;ax^{-\beta};q) \qquad (4.4)$$

Proof: Using (3.1) and q-fractional integral formula (2.16), we obtained

$$K \equiv \left(D_{q,-}^{\alpha}\left[t^{\alpha-\gamma}\ {}_{p}^{\beta}M_{r}^{\gamma}((a_j;q)_1^p;(b_j;q)_1^r;at^{-\beta};q)\right]\right)(x)$$

$$= \left(\frac{-1}{q}\right)^k D_{q^{-1}}^{k}\ \left(I_{q,-}^{k-\alpha}\left[\, t^{\alpha-\gamma}\ \ {}_{p}^{\beta}M_{r}^{\gamma}((a_j;q)_1^p;(b_j;q)_1^r;at^{-\beta};q\,)\right]\right)(x)$$

$$= \frac{1}{\Gamma_q(k-\alpha)}\left(\frac{-1}{q}\right)^k D_{q^{-1}}^{k}\int_{qx}^{\infty} t^{k-\alpha-1}\,({}^{qx}/_{t}\,;q)_{k-\alpha-1}\,t^{\alpha-\gamma}$$
$$\times\sum_{k=0}^{\infty}\frac{(a_1;q)_k\ \ldots\ldots(a_p;q)_k}{(b_1;q)_k\ \ldots\ldots(b_r;q)_k\ (q;q)_k}\ \frac{a^k t^{-\beta k}}{\Gamma_q(\beta k+\gamma)}\ d_q t$$

$$= \frac{1}{\Gamma_q(k-\alpha)}\sum_{k=0}^{\infty}\frac{(a_1;q)_k\ \ldots\ldots(a_p;q)_k}{(b_1;q)_k\ \ldots\ldots(b_r;q)_k\ (q;q)_k}\ \frac{a^k}{\Gamma_q(\beta k+\gamma)}$$
$$\times\left(\frac{-1}{q}\right)^k D_{q^{-1}}^{k}\int_{qx}^{\infty} t^{k-\gamma-\beta k-1}\,({}^{qx}/_{t}\,;q)_{k-\alpha-1}\ d_q t$$

Using equation (2.3), we get

$$K = \frac{1}{\Gamma_q(k-\alpha)}\sum_{k=0}^{\infty}\frac{(a_1;q)_k\ \ldots\ldots(a_p;q)_k}{(b_1;q)_k\ \ldots\ldots(b_r;q)_k\ (q;q)_k}\ \frac{a^k}{\Gamma_q(\beta k+\gamma)}$$
$$\times\left(\frac{-1}{q}\right)^k D_{q^{-1}}^{k}\int_{qx}^{\infty} t^{k-\gamma-\beta k-1}\ \frac{\left(\frac{qx}{t};q\right)_\infty}{\left(\frac{q^{k-\alpha}x}{t};q\right)_\infty}\ d_q t$$

Substituting $t=\frac{qx}{\xi}\ \ \Rightarrow\ d_q t = \frac{-qx}{\xi(\xi q)}d_q\xi = \frac{-x}{\xi^2}\ d_q\xi$, we get

$$K = \frac{1}{\Gamma_q(k-\alpha)}\sum_{k=0}^{\infty}\frac{(a_1;q)_k\ \ldots\ldots(a_p;q)_k}{(b_1;q)_k\ \ldots\ldots(b_r;q)_k\ (q;q)_k}\ \frac{a^k}{\Gamma_q(\beta k+\gamma)}$$
$$\times\left(\frac{-1}{q}\right)^k\ D_{q^{-1}}^{k}\ x^{k-\gamma-\beta k}q^{k-\gamma-\beta k-1}\int_0^1 \xi^{-k+\gamma+\beta k-1}\ \frac{(\xi;q)_\infty}{(q^{k-\alpha-1}\xi;q)_\infty}\ d_q\xi$$

Replacing (ξ) by (μq), we get

$$K = \frac{1}{\Gamma_q(k-\alpha)}\sum_{k=0}^{\infty}\frac{(a_1;q)_k\ \ldots\ldots(a_p;q)_k}{(b_1;q)_k\ \ldots\ldots(b_r;q)_k\ (q;q)_k}\ \frac{a^k}{\Gamma_q(\beta k+\gamma)}$$
$$\times\left(\frac{-1}{q}\right)^k D_{q^{-1}}^{k}\ x^{k-\gamma-\beta k}\ q^{-1}\int_0^1 (\mu)^{-k+\gamma+\beta k-1}\ \frac{(\mu q;q)_\infty}{(q^{k-\alpha}\mu;q)_\infty}\ d_q(\mu)$$

Using equation (2.11), it takes the form

$$K = \frac{1}{\Gamma_q(k-\alpha)} \sum_{k=0}^{\infty} \frac{(a_1;q)_k \ldots (a_p;q)_k}{(b_1;q)_k \ldots (b_r;q)_k \, (q;q)_k} \frac{a^k}{\Gamma_q(\beta k+\gamma)} \times \left(\frac{-1}{q}\right)^k D_{q^{-1}}^k \, x^{k-\gamma-\beta k} \, q^{-1} B_q\,(-k+\gamma+\beta k, k-\alpha)$$

Now using the relation (2.12) and on simplification, the RHS of above equation reduce to

$$K = \frac{1}{q} \sum_{k=0}^{\infty} \frac{(a_1;q)_k \ldots (a_p;q)_k}{(b_1;q)_k \ldots (b_r;q)_k \, (q;q)_k} \frac{a^k}{\Gamma_q(\beta k+\gamma)} \left(\frac{-1}{q}\right)^k D_{q^{-1}}^k \, x^{k-\gamma-\beta k} \times \frac{\Gamma_q(-k+\gamma+\beta k)}{\Gamma_q(\gamma+\beta k-\alpha)}$$

Also using equation (2.6), we get

$$K = \frac{1}{q} \sum_{k=0}^{\infty} \frac{(a_1;q)_k \ldots (a_p;q)_k}{(b_1;q)_k \ldots (b_r;q)_k \, (q;q)_k} \frac{a^k}{\Gamma_q(\beta k+\gamma)} \left(\frac{-1}{q}\right)^k q^k D_q^k \left(\frac{x^{k-\gamma-\beta k}}{q^k}\right) \times \frac{\Gamma_q(-k+\gamma+\beta k)}{\Gamma_q(\gamma+\beta k-\alpha)}$$

$$= \frac{1}{q^{k+1}} \, (-1)^k x^{-\gamma-\beta k} \sum_{k=0}^{\infty} \frac{(a_1;q)_k \ldots (a_p;q)_k}{(b_1;q)_k \ldots (b_r;q)_k \, (q;q)_k} \frac{a^k}{\Gamma_q(\beta k+\gamma)} \times \frac{\Gamma_q(k-\gamma-\beta k+1)}{\Gamma_q(-\gamma-\beta k+1)} \frac{\Gamma_q(-k+\gamma+\beta k)}{\Gamma_q(\gamma+\beta k-\alpha)}$$

$$= \frac{1}{q^{k+1}} \, (-1)^k x^{-\gamma} \sum_{k=0}^{\infty} \frac{(a_1;q)_k \ldots (a_p;q)_k}{(b_1;q)_k \ldots (b_r;q)_k \, (q;q)_k} \frac{(-\gamma-\beta k+1;q)_k}{(-1)^k(1-\beta k-\gamma;q)_k} \frac{(ax^{-\beta})^k}{\Gamma_q(\beta k+\gamma-\alpha)}$$

$$= \frac{1}{q^{k+1}} \, x^{-\gamma} \sum_{k=0}^{\infty} \frac{(a_1;q)_k \ldots (a_p;q)_k}{(b_1;q)_k \ldots (b_r;q)_k \, (q;q)_k} \frac{(ax^{-\beta})^k}{\Gamma_q(\beta k+\gamma-\alpha)}$$

$$= \frac{1}{q^{k+1}} \, x^{-\gamma} \, {}_p^{\beta}M_r^{\gamma-\alpha}((a_j;q)_1^p; (b_j;q)_1^r, ax^{-\beta}; q)$$

This completes the proof of (4.4).

Remark 4:- If we put $p = r = 1,\ a = \delta \in C,\ b = 1$ in (4.4), we shall obtain

$$\left(D_{q,-}^{\alpha} \left[t^{\alpha-\gamma} E_{\beta,\gamma}^{\delta}\left(at^{-\beta};q\right)\right]\right)(x) = \frac{1}{q^{k+1}} \, x^{-\gamma} \, E_{\beta,\gamma-\alpha}^{\delta}(ax^{-\beta};q)$$

5. Special cases:

In this section, we discuss some of the special cases of the main results established in the previous section, if we take $q = 1$ in the theorems (1), (2), (3) and (4), we have well-known results reported in [8] as follows

I. $\left(I_{0+}^{\alpha}\left[t^{\gamma-1}\ {}_{p}^{\beta}M_{r}^{\gamma}((a_j)_1^p;(b_j)_1^r;at^{\beta})\right]\right) = x^{\alpha+\gamma-1}\ {}_{p}^{\beta}M_{r}^{\alpha+\gamma}((a_j)_1^p;(b_j)_1^r;ax^{\beta})$

II. $\left(I_{-}^{\alpha}\left[t^{-\alpha-\gamma}\ {}_{p}^{\beta}M_{r}^{\gamma}\left((a_j)_1^p;(b_j)_1^r;at^{-\beta}\right)\right]\right)(x) = x^{-\gamma}\ {}_{p}^{\beta}M_{r}^{\alpha+\gamma}((a_j)_1^p;(b_j)_1^r;ax^{-\beta})$

III. $\left(D_{0+}^{\alpha}\left[t^{\gamma-1}\ {}_{p}^{\beta}M_{r}^{\gamma}((a_j)_1^p;(b_j)_1^r;at^{\beta})\right]\right)(x) = x^{\gamma-\alpha-1}\ {}_{p}^{\beta}M_{r}^{\gamma-\alpha}((a_j)_1^p;(b_j)_1^r;ax^{\beta})$

IV. $\left(D_{-}^{\alpha}\left[t^{\alpha-\gamma}\ {}_{p}^{\beta}M_{r}^{\gamma}\left((a_j)_1^p;(b_j)_1^r;at^{-\beta}\right)\right]\right)(x) = x^{-\gamma}\ {}_{p}^{\beta}M_{r}^{\gamma-\alpha}((a_j)_1^p;(b_j)_1^r;ax^{-\beta})$

6. Conclusion:

The results proved in this paper give some contributions to the theory of the q-fractional calculus, especially q-analogue of generalized M-series. The results proved in this paper appear to be new and likely to have useful applications to a wide range of problems of mathematics, statistics and physical sciences.

REFERENCES

1) Gasper G. and Rahman M. (2004): Basic Hypergeometric Series 2nd ed., Cambridge University Press, Cambridge.
2) Mansour Z.S.I. (2016): On fractional q-Sturm-Liouville problems, J. Fixed Point Theory Appl., doi: 10.1007/s11784-016-0331-y.
3) Bilender P. Allahverdiev, Huseyin Tuna (2017): Spectral Analysis of q-Fractional Sturm –Liouville Operators. Electronic Journal of Differential Equations, Vol. 2017, No.136, pp. 1 – 17.
4) Sharma M. (2008): Fractional integration and fractional differentiation of the M-series. Fract. Calc. Appl. Anal., 11(2), 187-192.
5) Sharma M. and Jain R. (2009): A note on a generalized M-series as a special function of fractional calculus. Fract. Calc. Appl. Anal., 12(4), 449-452.
6) Sunil Dutt Purohit and Faruk Ucar (2018): An application of q-Sumudu transform for fractional q-kinetic equation, Turk J Math 42:726-734.
7) Sharma S.K. and Jain R. (2014): On some properties of generalized q-Mittag Leffler Function, Mathematica Aeterna, Vol. 4, no. 6, 613-619.
8) Amit Chouhan and Satish Saraswat (2012): Certain properties of fractional calculus operators associated with M-series, Mathematical Sciences, Vol. 22, pp. 27-32.
9) Rajkovic P.M., Marinkovic S.D. and Stankovic M.S. (2007): Fractional integrals and derivatives in q-calculus. Applicable Analysis and Discrete Mathematics. 1,311 – 323.
10) Saxena, R.K., Rajendra Kumar (1995): A basic Analogue of the Generalized H-function Le Mathematiche Vol.L- Fase II 263-271.
11) Annaby M.H., Mansour Z.S. (2012): q-Fractional Calculus and Equations. Springer, 2056.