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Marichev-Saigo Maeda Fractional Calculus Operators and the Image Formulas of the Product of Generalized Gauss Hypergeometric Function and the K-Function

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Abstract
In this paper we will implement the generalized fractional operators involving the Appell's $F_3(.)$ Function due to Marichev-Saego Maeda to the product of the generalized Gauss hypergeometric function and the K-Function and will establish the image formulas of the product of the generalized Gauss hypergeometric function and the K-function in terms of the generalized Wright hypergeometric function. Special cases of the results are also mentioned in the concluding section of the paper.

Keywords
Generalized Gauss hypergeometric function, K-Function, Marichev-Saego Maeda fractional operators, Fox Wright function.

1. Introduction

The fractional calculus is the field of applied mathematics that deals with derivatives and integrals of arbitrary orders. Fractional calculus came into existence almost three centuries ago but it was not so popular in its initial phase mainly due to lack of applications. Recently it has gained a lot of significance and has drawn the attention of large number of mathematicians in view of its tremendous applicability in various sub-fields of applicable mathematical analysis. The mathematicians like Agarwal ([2]- [4]), Agarwal and Jain [5], Kalla and Saxena [8], Kilbas [9], Purohit and Kalla [15] and Saigo ([18]- [19]),Suthar et.al ([21]-[22]), Miller et al.[10] so forth have studied, in depth, the properties, applications, and different extensions of various operators of fractional calculus. The computation of fractional derivatives and the fractional integrals of special functions of one and more variables is important from the point of view of the usefulness of these results in the evaluation of generalized integrals and the solution of differential and integral equations.

Here, we aim at establishing the image formulas of the product of Generalized Gauss hypergeometric function and the K-function by applying the generalized fractional integral and the differential operators given by Marichev-Saigo Maeda.

2. Mathematical Preliminaries

Def 1. The generalized Beta function has been given by Özergin et al. [11] in their paper and has been defined as

$$B_p^{(\delta,\zeta)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,{}_1F_1\left(\delta,\zeta;\frac{-p}{t(1-t)}\right)dt \qquad (2.1)$$

$$\Re(p) \geq 0; \min\big(\Re(x),\Re(y),\Re(\delta),\Re(\zeta)\big) > 0 \qquad (2.2)$$

Def 2. Özergin et al with the introduction of generalized Beta function also studied and gave the definition of the family of generalized Gauss hypergeometric function and confluent hypergeometric functions [12] in the following way

$$F_p^{(\delta,\zeta)}(a,b;c;x) = \sum_{r=0}^{\infty}(a)_r \frac{B_p^{(\delta,\zeta)}(b+r,c-b)}{B(b,c-b)}\frac{x^r}{r!} \qquad (2.3)$$

$$ {}_1F_1^{(\delta,\zeta;p)}(b;c;x) = \sum_{r=0}^{\infty} \frac{B_p^{(\delta,\zeta)}(b+r,c-b)}{B(b,c-b)} \frac{x^r}{r!} \qquad (2.4) $$

Where $|x| < 1, \min(\Re(\delta),\Re(\zeta) > 0), \Re(c) > \Re(b) > 0 \;\&\Re(p) \geq 0$

$$ (\propto)_r = \begin{cases} 1 & r = 0 \\ \propto(\propto+1)(\propto+2)\ldots(\propto+r-1) & r > 0 \end{cases} \qquad (2.5) $$

provided $\alpha \neq 0$

$$ (\propto)_r = \frac{\Gamma(\propto+r)}{\Gamma(\propto)} (\propto \epsilon\, \mathbb{C} \setminus \mathbb{Z}_0^-) $$

$\mathbb{Z}_0^-$ denotes the set of non-positive integers.

Def 3. The generalized Fox- Wright function ${}_p\psi_q$ was introduced by Wright [23] and has been given by the series

$$ {}_p\psi_q = {}_p\psi_q \left\{ \begin{matrix} (a_1,A_1),\ldots(a_p,A_p); \\ (b_1,B_1),\ldots(b_q,B_q); \end{matrix} x \right\} = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i+nA_i)}{\prod_{j=1}^{q} \Gamma(b_j+nB_j)} \frac{x^n}{n!} \qquad (2.6) $$

where $\Gamma(x)$is the Euler gamma function.

where $x, a_i, b_j \in \mathbb{C}$; $A_i, B_j \in \Re$; $A_i \neq 0, B_j \neq 0; i = 1, \ldots, p; \quad j = 1, \ldots, q$

This function is known as generalized Wright function for all values of x. the conditions for its existence are as follows:

$$ 1 + \left(\sum_{j=1}^{q} B_j\right) - \left(\sum_{i=1}^{p} A_i\right) \geq 0 \qquad (2.7) $$

which has been derived by Wright E. M. [24].Several Properties of the generalized Wright have been studied and investigated by Kilbas et al. [1].

Def 4. The K-function introduced by Sharma [20] has been defined as follows:

$$ {}_pK_q^{\mu,\xi;\upsilon}(a_1,\ldots,a_p;b_1,\ldots,b_q;x) = {}_pK_q^{\mu,\xi;\upsilon}(x) $$

$$ = \sum_{n=0}^{\infty} \frac{(a_1)_n,(a_2)_n,\ldots,(a_p)_n(\upsilon)_n}{(b_1)_n,(b_2)_n,\ldots,(b_q)_n\Gamma(\mu n+\xi)} \frac{x^n}{n!} \qquad (2.8) $$

Where $\mu, \xi, \upsilon \in \mathbb{C}, \Re(\mu) > 0$

$(a_i)_n$ $(i = 1,2,\ldots p)$ and $(b_j)_n (j = 1,2,\ldots q)$ are the Pochhammer symbols. The series (2.8) is defined when none of the parameters $(b_j)_n, j = 1,2,\ldots,q$, is a negative integer or zero. If any numerator parameter $(a_i)_n$ is a negative integer or zero, then the series terminates to a polynomial in x. From the ratio test it is evident that the series is convergent for all x if $p > q + 1$.

Let $\upsilon = \sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j$, then for $p = q + 1$, the series converges absolutely for $|x| = 1$ if $\Re(\upsilon) < 0$, the series is conditionally convergent for $x = -1$ if $0 \leq \Re(\upsilon) < 1$

and the series is divergent for $|x| = 1$ if $1 \leq \Re(\upsilon)$.

The relation between the Wright generalized hypergeometric function and the K-function is given by [7]:

$$ {}_pK_q^{\mu,\xi;\upsilon}\left((a_i)_1^p;(b_j)_1^q\right) = \frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)} {}_{p+1}\psi_{q+1} \left\{ \begin{matrix} (a_1,1),\ldots(a_p,1),(\upsilon,1); \\ (b_1,1),\ldots(b_q,1),(\xi,\mu); \end{matrix} x \right\} \qquad (2.9) $$

Def 5. Saigo and Maeda [16] introduced the following generalized fractional and differential operators of any complex order with Appell $F_3(.)$ function in the kernel, as follows:

Let $\alpha,\alpha',\beta,\beta',\gamma \in \mathbb{C}$ and $x > 0$, then the generalized fractional calculus operators (the Marichev-Saigo-Maeda operators) involving the Appell function, or Horn's F_3 -function are defined by the following equations:

$$\left(I_{0+}^{\alpha,\alpha\prime,\beta,\beta\prime,\gamma} f\right)(x) = \frac{x}{\Gamma(\gamma)}^{-\alpha} \int_0^x (x-t)^{\gamma-1} t^{-\alpha\prime}$$

$$\times F_3\left(\alpha,\alpha',\beta,\beta';\ \gamma; 1-\frac{t}{x}, 1-\frac{x}{t}\right) f(t)dt, \quad (\Re(\gamma) > 0) \tag{2.10}$$

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(\frac{d}{dx}\right)^{k} \left(I_{0+}^{\alpha,\alpha',\beta+k,\beta',\gamma+k} f\right)(x) \tag{2.11}$$

$$(\Re(\gamma) \le 0 ;\ k = [-\Re(\gamma) + 1])$$

$$\left(I_{0-}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \frac{x}{\Gamma(\gamma)}^{-\alpha\prime} \int_x^\infty (t-x)^{\gamma-1} t^{-\alpha}$$

$$\times F_3\left(\alpha,\alpha',\beta,\beta';\ \gamma; 1-\frac{x}{t}, 1-\frac{t}{x}\right) f(t)dt, (\Re(\gamma) > 0 \tag{2.12}$$

$$\left(I_{0-}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(-\frac{d}{dx}\right)^{k} \left(I_{0-}^{\alpha,\alpha',\beta,\beta'+k,\gamma+k} f\right)(x), \tag{2.13}$$

$$(\Re(\gamma) \le 0 ;\ k = [-\Re(\gamma) + 1])$$

and

$$\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(I_{0+}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma} f\right)(x) \tag{2.14}$$

$$= \left(\frac{d}{dx}\right)^{k} \left(I_{0+}^{-\alpha',-\alpha,-\beta'+k,-\beta,-\gamma+k} f\right)(x), \quad (\Re(\gamma) > 0 ;\ k = [\Re(\gamma) + 1])$$

$$\left(D_{0-}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(I_{0-}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma} f\right)(x) \tag{2.15}$$

$$= \left(-\frac{d}{dx}\right)^{k} \left(I_{0-}^{-\alpha',-\alpha,-\beta',-\beta+k,-\gamma+k} f\right)(x), \quad (\Re(\gamma) > 0);\ k = [\Re(\gamma) + 1]$$

where the function $F_3(.)$ denotes the Appell function which has been introduced by Appell and Kampe de Feriet [6].

Following Saigo and Maeda [17], the image formulas for a power function, under operators (2.10) and (2.12) are given by:

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} x^{\rho-1}\right)(x) = x^{\rho-\alpha-\alpha'+\gamma-1}$$

$$\times \left[\frac{\Gamma(\rho)\Gamma(\rho+\gamma-\alpha-\alpha'-\beta)\Gamma(\rho+\beta'-\alpha')}{\Gamma(\rho+\beta')\Gamma(\rho+\gamma-\alpha-\alpha')\Gamma(\rho+\gamma-\alpha'-\beta)}\right] \tag{2.16}$$

where

$\Re(\rho) > max\ \{0, \Re(\alpha+\alpha'+\beta-\gamma), \Re(\alpha'-\beta')\}$ and $\Re(\gamma) > 0$.

$$\left(I_{0-}^{\alpha,\alpha',\beta,\beta',\gamma} x^{\rho-1}\right)(x) = x^{\rho+\gamma-\alpha-\alpha'-1}$$

$$\times \left[\frac{\Gamma(1-\rho-\beta)\Gamma(1-\rho-\gamma+\alpha+\alpha')\Gamma(1-\rho+\beta'+\alpha-\gamma)}{\Gamma(1-\rho)\Gamma(1-\rho+\beta'-\gamma+\alpha+\alpha')\Gamma(1-\rho+\alpha-\beta)}\right] \tag{2.17}$$

where

$$\Re(\rho) < 1 + min\,\{\Re(-\beta), \Re(\alpha+\alpha'-\gamma), \Re(\alpha+\beta'-\gamma)\}\; and\; \Re(\gamma) > 0.$$

3. Main results

Throughout all the theorems we will assume that $x > 0$, $\alpha, \alpha', \beta, \beta', \gamma, \rho, \mu, \xi, \upsilon, \delta, \zeta, p \in \mathbb{C}, \Re(\mu) > 0, a \in \mathbb{C}$, $\min(\Re(\delta), \Re(\zeta) > 0)$, $\Re(c) > \Re(b) > 0$ & $\Re(p) \geq 0$. Further we will assume that the constants satisfy the conditions $a_i, b_j \in \mathbb{C}, A_i, B_j \in \Re\ (A_i, B_j \neq 0, i = 1,2, \ldots, p, j = 1,2, \ldots, q.)$, and also the condition (2.7) is satisfied.

Theorem 1.1 Let $\Re(\xi) > 0, \Re(\upsilon) > 0$, then the fractional integral $I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}$ of the product of the K-function and the generalized Gauss hypergeometric function exists under the conditions $\Re(\gamma) > 0, \Re(\lambda) > 0, \Re(\rho + r) > \max\{0, \Re(\alpha + \alpha' + \beta - \gamma), \Re(\alpha' - \beta')\}$ and is given by

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\, t^{\rho-1} F_p^{(\delta,\zeta)}(a,b;c;t)\, {}^{\mu,\xi;\upsilon}_{\;\;p}K_q\left(bt^{\lambda}\right)\right)(x)$$
$$= \frac{x^{\rho-\alpha-\alpha'+\gamma-1}}{\Gamma\upsilon} F_p^{(\delta,\zeta)}(a,b;c;x) \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}$$
$$\times\, {}_{p+4}\psi_{q+4}\left[\begin{matrix}(a_1,1),\ldots(a_p,1),(\upsilon,1),(\rho+r,\lambda),(\rho+\gamma-\alpha-\alpha'-\beta+r,\lambda),\\ (b_1,1),\ldots(b_q,1),(\xi,\mu),(\rho+\beta'+r,\lambda),(\rho+\gamma-\alpha-\alpha'+r,\lambda),\end{matrix}\right.$$
$$\left.\begin{matrix}(\rho+\beta'-\alpha'+r,\lambda);\\ (\rho+\gamma-\beta-\alpha'+r,\lambda);\end{matrix}\, bx^{\lambda}\right] \tag{3.1}$$

Proof. Taking the LHS of (3.1) as $\mathcal{J}$ and using (2.3) and (2.8) we get

$$\mathcal{J} = \left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\, t^{\rho-1} \sum_{r=0}^{\infty} (a)_r \frac{B_p^{(\delta,\zeta)}(b+r, c-b)}{B(b,c-b)} \frac{t^r}{r!}\right.$$
$$\left.\times \sum_{n=0}^{\infty} \frac{(a_1)_n, (a_2)_n, \ldots, (a_p)_n (\upsilon)_n}{(b_1)_n, (b_2)_n, \ldots, (b_q)_n \Gamma(\mu n+\xi)} \frac{(bt^{\lambda})^n}{n!}\right)(x) \tag{3.2}$$

$$\mathcal{J} = \sum_{r=0}^{\infty} (a)_r \frac{B_p^{(\delta,\zeta)}(b+r, c-b)}{B(b,c-b)} \frac{1}{r!} \sum_{n=0}^{\infty} \frac{(a_1)_n, (a_2)_n, \ldots, (a_p)_n (\upsilon)_n}{(b_1)_n, (b_2)_n, \ldots, (b_q)_n \Gamma(\mu n+\xi)} \frac{b^n}{n!}$$
$$\times \left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\rho+r+\lambda n-1}\right)\right)(x) \tag{3.3}$$

Applying (2.16) on (3.3), we get

$$\mathcal{J} = x^{\rho-\alpha-\alpha'+\gamma-1}$$
$$\times \sum_{r=0}^{\infty} (a)_r \frac{B_p^{(\delta,\zeta)}(b+r, c-b)}{B(b,c-b)} \frac{x^r}{r!} \sum_{n=0}^{\infty} \frac{(a_1)_n, (a_2)_n, \ldots, (a_p)_n (\upsilon)_n}{(b_1)_n, (b_2)_n, \ldots, (b_q)_n \Gamma(\mu n+\xi)} \frac{(bx^{\lambda})^n}{n!}$$

$$\times \left[\frac{\Gamma(\rho+r+\lambda n)\ \Gamma(\rho+\gamma-\alpha-\alpha'-\beta+r+\lambda n)\, \Gamma(\rho+\beta'-\alpha'+r+\lambda n)}{\Gamma(\rho+\beta'+r+\lambda n)\ \Gamma(\rho+\gamma-\alpha-\alpha'+r+\lambda n)\ \Gamma(\rho+\gamma-\beta-\alpha'+r+\lambda n)}\right] \tag{3.4}$$

Using (2.9) in (3.4), we get

$$\mathcal{J} = \frac{x^{\rho-\alpha-\alpha'+\gamma-1}}{\Gamma\upsilon} \sum_{r=0}^{\infty} (a)_r \frac{B_p^{(\delta,\zeta)}(b+r, c-b)}{B(b,c-b)} \frac{x^r}{r!} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}$$

$$\times {}_{p+1}\psi_{q+1}\left\{\begin{matrix}(a_1,1),\dots(a_p,1),(\upsilon,1);\\(b_1,1),\dots(b_q,1),(\xi,\mu);\end{matrix} bx^{\lambda}\right\}$$

$$\times\left[\frac{\Gamma(\rho+r+\lambda n)\ \Gamma(\rho+\gamma-\alpha-\alpha'-\beta+r+\lambda n)\ \Gamma(\rho+\beta'-\alpha'+r+\lambda n)}{\Gamma(\rho+\beta'+r+\lambda n)\ \Gamma(\rho+\gamma-\alpha-\alpha'+r+\lambda n)\ \Gamma(\rho+\gamma-\beta-\alpha'+r+\lambda n)}\right] \quad (3.5)$$

Interpreting further and we arrived at the result.

Theorem 1.2 Let $\Re(\xi)>0, \Re(\upsilon)>0$, then the fractional integral $I_{0-}^{\alpha,\alpha',\beta,\beta',\gamma}$ of the product of the K-function and the generalized Gauss hypergeometric function exists under the conditions $\Re(\gamma)>0, \Re(\lambda)>0, \Re(1-\gamma-\rho-r)<1+\min\{\Re(-\beta), \Re(\alpha+\alpha'-\gamma), \Re(\alpha+\beta'-\gamma)\}$ and is given by

$$\left(I_{0-}^{\alpha,\alpha',\beta,\beta',\gamma}\, t^{-\gamma-\rho} F_p^{(\delta,\zeta)}\left(a,b;c;\frac{1}{t}\right).\, {}_p^{\mu,\xi,\upsilon}K_q\left(bt^{-\lambda}\right)\right)(x)$$

$$=\frac{x^{-\rho-\alpha-\alpha'}}{\Gamma\upsilon} F_p^{(\delta,\zeta)}\left(a,b;c;\frac{1}{x}\right)\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}$$

$$\times {}_{p+4}\psi_{q+4}\left[\begin{matrix}(a_1,1),\dots(a_p,1),(\upsilon,1),(\rho+\alpha+\beta'+r,\lambda),(\rho+\alpha+\alpha'+r,\lambda),\\(b_1,1),\dots(b_q,1),(\xi,\mu),(\rho+\alpha+\alpha'+\beta'+r,\lambda),(\rho+\gamma+r,\lambda),\end{matrix}\right.$$

$$\left.\begin{matrix}(\rho-\beta+\gamma+r,\lambda);\\(\rho+\alpha-\beta+\gamma+r,\lambda);\end{matrix} bx^{-\lambda}\right] \quad (3.6)$$

Proof. Taking the LHS of (3.6) as $\mathcal{I}$ and using (2.3) and (2.8) we get

$$\mathcal{I}=\left(I_{0-}^{\alpha,\alpha',\beta,\beta',\gamma}\, t^{-\gamma-\rho}\sum_{r=0}^{\infty}(a)_r\frac{B_p^{(\delta,\zeta)}(b+r,c-b)}{B(b,c-b)}\frac{1}{t^r r!}\right.$$

$$\left.\times\sum_{n=0}^{\infty}\frac{(a_1)_n,(a_2)_n,\dots,(a_p)_n(\upsilon)_n}{(b_1)_n,(b_2)_n,\dots,(b_q)_n\Gamma(\mu n+\xi)}\frac{(bt^{-\lambda})^n}{n!}\right)(x) \quad (3.7)$$

$$\mathcal{I}=\sum_{r=0}^{\infty}(a)_r\frac{B_p^{(\delta,\zeta)}(b+r,c-b)}{B(b,c-b)}\frac{1}{r!}\sum_{n=0}^{\infty}\frac{(a_1)_n,(a_2)_n,\dots,(a_p)_n(\upsilon)_n}{(b_1)_n,(b_2)_n,\dots,(b_q)_n\Gamma(\mu n+\xi)}\frac{b^n}{n!}$$

$$\times\left(\mathrm{I}_{0-}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{-\gamma-\rho-r-\lambda n}\right)\right)(x) \quad (3.8)$$

Applying (2.17) on (3.8), we get

$$\mathcal{I}=x^{-\rho-\alpha-\alpha'}$$

$$\times\sum_{r=0}^{\infty}(a)_r\frac{B_p^{(\delta,\zeta)}(b+r,c-b)}{B(b,c-b)}\frac{1}{x^r r!}\sum_{n=0}^{\infty}\frac{(a_1)_n,(a_2)_n,\dots,(a_p)_n(\upsilon)_n}{(b_1)_n,(b_2)_n,\dots,(b_q)_n\Gamma(\mu n+\xi)}\frac{(bx^{-\lambda})^n}{n!}$$

$$\times\left[\frac{\Gamma(\rho+\alpha+\alpha'+r+\lambda n)\ \Gamma(\rho+\alpha+\beta'+r+\lambda n)\ \Gamma(\rho-\beta+\gamma+r+\lambda n)}{\Gamma(\rho+\alpha+\alpha'+\beta'+r+\lambda n)\ \Gamma(\rho+\gamma+r+\lambda n)\ \Gamma(\rho+\alpha-\beta+\gamma+r+\lambda n)}\right] \quad (3.9)$$

Using (2.9) in (3.9), we get

$$\mathcal{I}=\frac{x^{-\rho-\alpha-\alpha'}}{\Gamma\upsilon}\sum_{r=0}^{\infty}(a)_r\frac{B_p^{(\delta,\zeta)}(b+r,c-b)}{B(b,c-b)}\frac{1}{x^r r!}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}$$

$$\times {}_{p+1}\psi_{q+1}\left\{\begin{matrix}(a_1,1),\dots(a_p,1),(\upsilon,1);\\(b_1,1),\dots(b_q,1),(\xi,\mu);\end{matrix} bx^{-\lambda}\right\}$$

$$\times\left[\frac{\Gamma(\rho+\alpha+\alpha'+r+\lambda n)\Gamma(\rho+\alpha+\beta'+r+\lambda n)\Gamma(\rho-\beta+\gamma+r+\lambda n)}{\Gamma(\rho+\alpha+\alpha'+\beta'+r+\lambda n)\ \Gamma(\rho+\gamma+r+\lambda n)\Gamma(\rho+\alpha-\beta+\gamma+r+\lambda n)}\right] \quad (3.10)$$

Interpreting further and we arrived at the result.

Theorem 1.3 Let $\Re(\xi)>0, \Re(v)>0$, then the fractional differential $D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}$ of the product of the K-function and the generalized Gauss hypergeometric function exists under the conditions $\Re(\gamma)>0, \Re(\lambda)>0, \Re(\rho+r)>max\{0, \Re(\gamma-\alpha-\alpha'-\beta'), \Re(\beta-\alpha)\}$ and is given by

$$\left(D_{0+}^{\alpha,\alpha\prime,\beta,\beta\prime,\gamma}t^{\rho-1}F_p^{(\delta,\zeta)}(a,b;c;t).\,{}^{\mu,\xi,v}_{p}K_q(bt^{\lambda})\right)(x)$$

$$=\frac{x^{\rho+\alpha+\alpha\prime-\gamma-1}}{\Gamma v}F_p^{(\delta,\zeta)}(a,b;c;x)\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}$$

$$\times\ {}_{p+4}\psi_{q+4}\left[\begin{matrix}(a_1,1),\ldots(a_p,1),(v,1),(\rho+r,\lambda),(\rho-\gamma+\alpha+\alpha'+\beta'+r,\lambda),\\(b_1,1),\ldots(b_q,1),(\xi,\mu),(\rho-\beta+r,\lambda),(\rho-\gamma+\alpha+\alpha'+r,\lambda),\end{matrix}\right.$$

$$\left.\begin{matrix}(\rho-\beta+\alpha+r,\lambda);\\(\rho-\gamma+\beta'+\alpha+r,\lambda);\end{matrix}bx^{\lambda}\right] \quad (3.11)$$

Proof. Taking the LHS of (3.11) as $\mathcal{I}$ and using (2.3) and (2.8) we get

$$\mathcal{I}=\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}\sum_{r=0}^{\infty}(a)_r\frac{B_p^{(\delta,\zeta)}(b+r,c-b)}{B(b,c-b)}\frac{t^r}{r!}\right.$$

$$\left.\times\sum_{n=0}^{\infty}\frac{(a_1)_n,(a_2)_n,\ldots,(a_p)_n(v)_n}{(b_1)_n,(b_2)_n,\ldots,(b_q)_n\Gamma(\mu n+\xi)}\frac{(bt^{\lambda})^n}{n!}\right)(x) \quad (3.12)$$

Using (2.14) we get

$$\mathcal{I}=\sum_{r=0}^{\infty}(a)_r\frac{B_p^{(\delta,\zeta)}(b+r,c-b)}{B(b,c-b)}\frac{1}{r!}\sum_{n=0}^{\infty}\frac{(a_1)_n,(a_2)_n,\ldots,(a_p)_n(v)_n}{(b_1)_n,(b_2)_n,\ldots,(b_q)_n\Gamma(\mu n+\xi)}\frac{b^n}{n!}$$

$$\times\left(I_{0+}^{-\alpha',-\alpha,-\alpha',-\beta,-\gamma}(t^{\rho+r+\lambda n-1})\right)(x) \quad (3.13)$$

Applying (2.16) on (3.13), we get

$$\mathcal{I}=x^{\rho+\alpha+\alpha'-\gamma-1}$$

$$\times\sum_{r=0}^{\infty}(a)_r\frac{B_p^{(\delta,\zeta)}(b+r,c-b)}{B(b,c-b)}\frac{x^r}{r!}\sum_{n=0}^{\infty}\frac{(a_1)_n,(a_2)_n,\ldots,(a_p)_n(v)_n}{(b_1)_n,(b_2)_n,\ldots,(b_q)_n\Gamma(\mu n+\xi)}\frac{(bx^{\lambda})^n}{n!}$$

$$\times\left[\frac{\Gamma(\rho+r+\lambda n)\ \Gamma(\rho-\gamma+\alpha+\alpha'+\beta'+r+\lambda n)\Gamma(\rho-\beta+\alpha+r+\lambda n)}{\Gamma(\rho-\beta+r+\lambda n)\Gamma(\rho-\gamma+\alpha+\alpha'+r+\lambda n)\Gamma(\rho-\gamma+\beta'+\alpha+r+\lambda n)}\right] \quad (3.14)$$

Using (2.9) in (3.14), we get

$$\mathcal{I}=\frac{x^{\rho+\alpha+\alpha\prime-\gamma-1}}{\Gamma v}\sum_{r=0}^{\infty}(a)_r\frac{B_p^{(\delta,\zeta)}(b+r,c-b)}{B(b,c-b)}\frac{x^r}{r!}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}$$

$$\times\ {}_{p+1}\psi_{q+1}\left\{\begin{matrix}(a_1,1),\ldots(a_p,1),(v,1);\\(b_1,1),\ldots(b_q,1),(\xi,\mu);\end{matrix}bx^{\lambda}\right\}$$

$$\times\left[\frac{\Gamma(\rho+r+\lambda n)\Gamma(\rho-\gamma+\alpha+\alpha'+\beta'+r+\lambda n)\Gamma(\rho-\beta+\alpha+r+\lambda n)}{\Gamma(\rho-\beta+r+\lambda n)\Gamma(\rho-\gamma+\alpha+\alpha'+r+\lambda n)\Gamma(\rho-\gamma+\beta'+\alpha+r+\lambda n)}\right] \quad (3.15)$$

Interpreting further and we arrived at the result.

Theorem 1.4 Let $\Re(\xi)>0, \Re(v)>0$, then the fractional differential $D_{0-}^{\alpha,\alpha\prime,\beta,\beta\prime,\gamma}$ of the product of the K-function and the generalized Gauss hypergeometric function exists under the conditions $\Re(\gamma) > 0, \Re(\lambda) > 0, \Re(1-\gamma-r-\rho) < 1 + min\{\Re(-\beta), \Re(\alpha+\alpha'-\gamma), \Re(\alpha+\beta'-\gamma)\}$ and is given by

$$\left(D_{0-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\gamma-\rho} F_p^{(\delta,\zeta)}\left(a,b;c;\frac{1}{t}\right). {}^{\mu,\xi,\upsilon}_{p}K_q\left(bt^{-\lambda}\right)\right)(x)$$
$$= \frac{x^{-\rho+\alpha+\alpha\prime}}{\Gamma\upsilon} F_p^{(\delta,\zeta)}\left(a,b;c;\frac{1}{x}\right)\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}$$
$$\times {}_{p+4}\psi_{q+4}\left[\begin{matrix}(a_1,1),...(a_p,1),(\upsilon,1),(\rho-\alpha-\alpha'+r,\lambda),(\rho-\beta-\alpha'+r,\lambda),\\(b_1,1),...\ (b_q,1),(\xi,\mu),(\rho-\alpha-\alpha'-\beta+r,\lambda),(\rho-\gamma+r,\lambda),\end{matrix}\right.$$
$$\left.\begin{matrix}(\rho+\beta'-\gamma+r,\lambda);\\(\rho-\alpha'+\beta'-\gamma+r,\lambda);\end{matrix} bx^{-\lambda}\right] \tag{3.16}$$

Proof. Taking the LHS of (3.16) as $\mathcal{J}$ and using (2.3) and (2.8) we get

$$\mathcal{J} = \left(D_{0-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\gamma-\rho}\sum_{r=0}^{\infty}(a)_r\frac{B_p^{(\delta,\zeta)}(b+r,c-b)}{B(b,c-b)}\frac{1}{t^r r!}\right.$$
$$\left.\times\sum_{n=0}^{\infty}\frac{(a_1)_n,(a_2)_n,...,(a_p)_n(\upsilon)_n}{(b_1)_n,(b_2)_n,...,(b_q)_n\Gamma(\mu n+\xi)}\frac{(bt^{-\lambda})^n}{n!}\right)(x) \tag{3.17}$$

Using (2.15) we get

$$\mathcal{J} = \sum_{r=0}^{\infty}(a)_r\frac{B_p^{(\delta,\zeta)}(b+r,c-b)}{B(b,c-b)}\frac{1}{r!}\sum_{n=0}^{\infty}\frac{(a_1)_n,(a_2)_n,...,(a_p)_n(\upsilon)_n}{(b_1)_n,(b_2)_n,...,(b_q)_n\Gamma(\mu n+\xi)}\frac{b^n}{n!}$$
$$\times\left(I_{0-}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma}\left(t^{\gamma-\rho-r-\lambda n}\right)\right)(x) \tag{3.18}$$

Applying (2.17) on (3.18), we get

$$\mathcal{J} = x^{-\rho+\alpha+\alpha\prime}\sum_{r=0}^{\infty}(a)_r\frac{B_p^{(\delta,\zeta)}(b+r,c-b)}{B(b,c-b)}\frac{1}{x^r r!}$$
$$\times\sum_{n=0}^{\infty}\frac{(a_1)_n,(a_2)_n,...,(a_p)_n(\upsilon)_n}{(b_1)_n,(b_2)_n,...,(b_q)_n\Gamma(\mu n+\xi)}\frac{(bx^{-\lambda})^n}{n!}$$
$$\times\left[\frac{\Gamma(\rho-\alpha-\alpha'+r+\lambda n)\ \Gamma(\rho-\alpha'-\beta+r+\lambda n)\ \Gamma(\rho+\beta'-\gamma+r+\lambda n)}{\Gamma(\rho-\alpha-\alpha'-\beta+r+\lambda n)\ \Gamma(\rho-\gamma+r+\lambda n)\ \Gamma(\rho-\alpha'+\beta'-\gamma+r+\lambda n)}\right] \tag{3.19}$$

Using (2.9) in (3.19), we get

$$\mathcal{J} = \frac{x^{-\rho+\alpha+\alpha\prime}}{\Gamma\upsilon}\sum_{r=0}^{\infty}(a)_r\frac{B_p^{(\delta,\zeta)}(b+r,c-b)}{B(b,c-b)}\frac{1}{x^r r!}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}$$
$$\times {}_{p+1}\psi_{q+1}\left\{\begin{matrix}(a_1,1),...(a_p,1),(\upsilon,1);\\(b_1,1),...(b_q,1),(\xi,\mu);\end{matrix} bx^{-\lambda}\right\}$$
$$\times\left[\frac{\Gamma(\rho-\alpha-\alpha'+r+\lambda n)\ \Gamma(\rho-\alpha'-\beta+r+\lambda n)\ \Gamma(\rho+\beta'-\gamma+r+\lambda n)}{\Gamma(\rho-\alpha-\alpha'-\beta+r+\lambda n)\ \Gamma(\rho-\gamma+r+\lambda n)\ \Gamma(\rho-\alpha'+\beta'-\gamma+r+\lambda n)}\right] \tag{3.20}$$

Interpreting further and we arrived at the result.

Special cases

I. If $a_i = b_j = 0$ i.e, there is no upper and lower parameter in the K-function and also $\mu = \xi = \upsilon = 1, \lambda = 0, n = 0$, then ${}^{1,1;1}_{\ \ \ 0}K_0(\_;\_;x) = 1$

Therefore (3.5) becomes

$$\mathcal{I} = x^{\rho-\alpha-\alpha'+\gamma-1} \sum_{r=0}^{\infty} (a)_r \frac{B_p^{(\delta,\zeta)}(b+r, c-b)}{B(b, c-b)} \frac{x^r}{r!} \times \left[\frac{\Gamma(\rho+r)\ \Gamma(\rho+\gamma-\alpha-\alpha'-\beta+r)\, \Gamma(\rho+\beta'-\alpha'+r)}{\Gamma(\rho+\beta'+r)\ \Gamma(\rho+\gamma-\alpha-\alpha'+r)\ \Gamma(\rho+\gamma-\beta-\alpha'+r)} \right]$$

$$\mathcal{I} = x^{\rho-\alpha-\alpha'+\gamma-1} \sum_{r=0}^{\infty} (a)_r \frac{B_p^{(\delta,\zeta)}(b+r, c-b)}{B(b, c-b)} \times \left[\frac{\Gamma(\rho)\ \Gamma(\rho+\gamma-\alpha-\alpha'-\beta)\, \Gamma(\rho+\beta'-\alpha')}{\Gamma(\rho+\beta')\ \Gamma(\rho+\gamma-\alpha-\alpha')\ \Gamma(\rho+\gamma-\beta-\alpha')} \right] \times \frac{(\rho)_r (\rho+\gamma-\alpha-\alpha'-\beta)_r (\rho+\beta'-\alpha')_r}{(\rho+\beta')_r (\rho+\gamma-\alpha-\alpha')_r (\rho+\gamma-\beta-\alpha')_r} \frac{x^r}{r!}$$

$$\mathcal{I} = x^{\rho-\alpha-\alpha'+\gamma-1} \left[\frac{\Gamma(\rho)\ \Gamma(\rho+\gamma-\alpha-\alpha'-\beta)\, \Gamma(\rho+\beta'-\alpha')}{\Gamma(\rho+\beta')\ \Gamma(\rho+\gamma-\alpha-\alpha')\ \Gamma(\rho+\gamma-\beta-\alpha')} \right] F_p^{(\delta,\zeta)}(a, b; c; x)\ {}_3F_3 \left[\begin{matrix} \rho, \rho+\gamma-\alpha-\alpha'-\beta, \rho+\beta'-\alpha'; \\ \rho+\beta', \rho+\gamma-\alpha-\alpha', \rho+\gamma-\beta-\alpha'; \end{matrix} x \right]$$

which are the results considered by Praveen et al. [13]

II. If $a_i = b_j = 0$ i.e, there is no upper and lower parameter in the K-function and also $\mu = \xi = \upsilon = 1, \lambda = 0, n = 0$, then ${}^{1,1;1}_{\ \ \ 0}K_0(\_;\_;x) = 1$

Therefore (3.10) becomes

$$\mathcal{I} = x^{-\rho-\alpha-\alpha'} \sum_{r=0}^{\infty} (a)_r \frac{B_p^{(\delta,\zeta)}(b+r, c-b)}{B(b, c-b)} \frac{1}{x^r r!} \times \left[\frac{\Gamma(\rho+\alpha+\alpha'+r+\lambda n)\Gamma(\rho+\alpha+\beta'+r+\lambda n)\Gamma(\rho-\beta+\gamma+r+\lambda n)}{\Gamma(\rho+\alpha+\alpha'+\beta'+r+\lambda n)\ \Gamma(\rho+\gamma+r+\lambda n)\Gamma(\rho+\alpha-\beta+\gamma+r+\lambda n)} \right]$$

Replacing $-\gamma-\rho$ by $\rho-1$ and on simplifying further, we obtain the results considered by Praveen et al. [13]

III. If $a_i = b_j = 0$ i.e, there is no upper and lower parameters in the K-function and also $\mu = \xi = \upsilon = 1, \lambda = 0, n = 0$, then ${}^{1,1;1}_{\ \ \ 0}K_0(\_;\_;x) = 1$

Therefore (3.15) becomes

$$\mathcal{I} = x^{\rho+\alpha+\alpha'-\gamma-1} \sum_{r=0}^{\infty} (a)_r \frac{B_p^{(\delta,\zeta)}(b+r, c-b)}{B(b, c-b)} \frac{x^r}{r!} \times \left[\frac{\Gamma(\rho+r)\Gamma(\rho-\gamma+\alpha+\alpha'+\beta'+r)\Gamma(\rho-\beta+\alpha+r)}{\Gamma(\rho-\beta+r)\Gamma(\rho-\gamma+\alpha+\alpha'+r)\Gamma(\rho-\gamma+\beta'+\alpha+r)} \right]$$

On simplifying further, we obtain the results considered by Praveen et al. [14]

IV. If $a_i = b_j = 0$ i.e, there is no upper and lower parameter in the K-function and also $\mu = \xi = \upsilon = 1, \lambda = 0, n = 0$, then ${}^{1,1;1}_{\ \ \ 0}K_0(\_;\_;x) = 1$

Therefore (3.20) becomes

$$\mathcal{I} = x^{-\rho+\alpha+\alpha'} \sum_{r=0}^{\infty} (a)_r \frac{B_p^{(\delta,\zeta)}(b+r, c-b)}{B(b, c-b)} \frac{1}{x^r r!} \times \left[\frac{\Gamma(\rho-\alpha-\alpha'+r)\ \Gamma(\rho-\alpha'-\beta+r)\, \Gamma(\rho+\beta'-\gamma+r)}{\Gamma(\rho-\alpha-\alpha'-\beta+r)\ \Gamma(\rho-\gamma+r)\ \Gamma(\rho-\alpha'+\beta'-\gamma+r)} \right]$$

Replacing $\gamma-\rho$ by $\rho-1$ and on simplifying further, we obtain the results considered by Praveen et al. [14]

4. Conclusion

In this paper we have studied certain image formulas of the product of two functions; the K-function and the generalized Guass hypergeometric function involving the Marichev Saego-Maeda fractional operators and we have arrived at the function known as Fox's Wright-Function.

As the results obtained are quite general so we can reduce the general results involving Saego-Maeda operators to the corresponding special results by assigning the different values to different parameters involved in the general results. Thus these results obtained can be applied to the number of different problems involved in different fields of Mathematics and engineering sciences.

References

1. A.A. Kilbas, M.Saigo and J.J.Trujillo(2002), "On the Generalized Wright Function", Fract. Calc. Appl. Anal, **5** (4), 437-460.
2. Agarwal, P. (2012), "Further results on fractional calculus of Saigo operators", Appl. Appl. Math., 7(2), 585- 594.
3. Agarwal, P. (2012), "Generalized fractional integration of the H function", Matematiche (Catania), 67(2), 107-118.
4. Agarwal, P.(2013), "Fractional integration of the product of two multivariable H-function and a general class of polynomials", Advances in Applied Mathematics and Approximation Theory, Springer Proc. Math. Stat., Springer, New York, 41, 359-374.
5. Agarwal, P., Jain, S. (2011), "Further results on fractional calculus of Srivastava polynomials", Bull. Math. Anal. Appl., 3, (2), 167-174.
6. Appell, P. and Kampe de Feriet, J. (1926), "Functions Hypergeometriqueset Hyperspheriques Polynomesd' Hermite", Gauthier-Viliars, Paris.
7. D.L Suthar, Haile H. (2017), "Marichev-Saigo Integral Operators Involving the Product of K-Function and Multivariable Polynomials", Communications in Numerical Analysis 2017 (2) 101-108
8. Kalla, S.L., Saxena, R.K. (1969), "Integral operators involving hypergeometric functions", Math. Z., 108, 231-234.
9. Kilbas, A.A. (2005), "Fractional calculus of the generalized Wright function", Fract. Calc. Appl. Anal.,8, (2), 113-126.
10. Miller, K.S., Ross, B. (1993), "An Introduction to the Fractional Calculus and Fractional Differential Equations", A Wiley-Inter science Publication. John Wiley & Sons, Inc., New York.
11. Özergin (2011),"Some properties of hyper geometric functions", Eastern Mediterranean University, North Cyprus, 2011 Ph.D. thesis.
12. Özergin, M.A.Özarslan, A. Altin (2011), "Extension of gamma, beta and hyper-geometric functions", J.Comput. Appl. Math. 235, 4601–4610.
13. Praveen A., Junesang Choi (2016), "Fractional calculus operators and their image formulas", J. Korean Math. Soc. 53(5),1183-1210
14. Praveen A., Mehar C., Erkinjon T.K. (2015), "Certain Image formulas of generalized hyper-geomtric functions", Applied Mathematics and Computation, 266,763-772.
15. Purohit, S.D., Kalla, S.L. (2011), "On fractional partial differential equations related to quantum Mechanics", J. Phys. A, 44, (4), Article ID 045202.
16. Saigo M. and Maeda N. (1996): "More Generalization of the Fractional Calculus. Transform Method and Special Function" Verna, Bulgaria, pp. 386-400.

17. Saigo M. and Maeda N. (1996), "More Generalization of the Fractional Calculus. Transform Method and Special Function" Verna, Bulgaria, pp. 386-400.
18. Saigo, M. (1978), "A remark on integral operators involving the Gauss hypergeometric functions", Math. Rep. Kyushu Univ., 11, (2), 135-143.
19. Saigo, M. (1979), "A certain boundary value problem for the Euler- Darboux equation", Math. Japon, 24(4), 377-385.
20. Sharma (2011), "Applications of Fractional Calculus Operators to Related Areas", ICSRS Publication, Gen. Math. Notes, Vol.7(1), November 2011, pp.33-40.
21. Suthar, D.L., Amsalu H. (2017), "Certain integrals associated with the generalized Bessel-Maitland Function", Appl. Appl. Math., 12, (2), 1002-1016.
22. Suthar, D.L., Habenom H. (2016), "Integrals involving generalized Bessel-Maitland Function:, J. Sci. Arts, 37, (4), 357-362.
23. Wright E. M. (1935): "The asymptotic expansion of the generalized hypergeometric functions" J. London Math. Soc., Vol. 10, pp. 286-293.
24. Wright E. M. (1940), "The asymptotic expansion of the generalized hypergeometric functions" Proc. London Math. Soc., 46(2),398-408.